Mathematical Reviews

Vol. 18, No. 7 JULY-AUGUST, 1957 pp. 551-630

FOUNDATIONS, THEORY OF SETS, LOGIC

Banaschewski, Bernhard. Hüllensysteme und Erweiterung von Quasi-Ordnungen. Z. Math. Logik Grundlagen Math. 2 (1956), 117–130.

Sei $(E; \leqq)$ eine Quasi-Ordnung ($\leqq$ ist reflexiv und transitiv); für $A \subseteq E$ sei

$$uA=\{x|x \in E,\ x \leqq A\},\ oA=\{x|x \in E,\ x \geqq A\}.$$

Das System $\mathcal{E}$ der ux $(x \in E)$ heisse das kanonische Bild von E in $PE=\{A|A \subseteq E\}$. Wenn H ein Hüllensystem auf E ist, das die X-Bedingung (Durchschnitt aller das Element $x \in E$ enthaltenden Mengen $X \in H$ gehört zu H) erfüllt, dann fragt man sich um solche Hüllensysteme H auf einer quasi-geordneter Menge, dessen Hüllenoperator Γ_H der Gleichung $\Gamma_H x=ux$ $(x \in E)$ befriedigt. Dies sind vollständige supremum-dichten Quasi-Erweiterungen von E (vollständig: jede Teilmenge hat Infimum und Supremum). A ist supremum-dicht im E, falls jeder Punkt aus E als Supremum einer Teilmenge von A darstellbar ist. Eine geordnete Menge G heisse eine Quasi-Erweiterung von $(E, \leqq)$, wenn es eine Abbildung $g: E \to G$ gibt so, dass $gx \leqq gy$ genau dann gilt, wenn $x \leqq y$). Sei $M \subseteq PE$; seien $\tilde{M}$, bzw. $\hat{M}$, bzw. M^*, das kleinste (volladditive, bzw. induktive) Hüllensystem $\supseteqq M$. Wenn E geordnet ist, so ist $\tilde{\mathcal{E}}$, bzw. $\hat{\mathcal{E}}$, im wesentlichen eindeutig bestimmt als die kleinste, bzw. grösste, vollständige supremum-dichte Erweiterung von E (Kor. 5, bzw. 6). Wenn E geordnet ist, dann ist $\mathcal{E}=\mathcal{E}^*$, falls E vollständig ist und jede nichtleere Teilmenge Maximal-Element besitzt (Th. 5). Es folgen noch Sätze über Ringe und Verbände.

Đ. Kurepa (Zagreb).

Bagemihl, F.; and Gillman, L. Some cofinality theorems on ordered sets. Fund. Math. 43 (1956), 178–184.

The main results are given below, the proofs of (1) and (2) being straightforward. (1) ω_α is cofinal with $\omega_\alpha\phi+\rho$ if and only if $\rho=0$ and either ϕ is isolated or $\mathrm{cf}(\phi)=\mathrm{cf}(\alpha)$. (2) For two ordinal numbers α and β let $T(\alpha, \beta)$ be the lexicographically ordered set of all sequences $t=(\tau_\xi)_{\xi<\omega_\beta}$, where $\tau_\xi<\omega_\alpha$, not every $\tau_\xi=0$, and only a finite number of τ_ξ are different from 0. Let $R(t)=\{x|t \leqq x,\ x \in T(\alpha, \beta)\}$. If $\beta \leqq \mathrm{cf}(\alpha)$, then a necessary and sufficient condition that $\beta=\mathrm{cf}(\alpha)$ is that there exists a set $M \subseteq T(\alpha, \beta)$, with $|M|=\aleph_\alpha$, such that $|M \cap R(t)|<\aleph_\alpha$ for every t in $T(\alpha, \beta)$. (3) Let $\alpha>0$. Then a necessary and sufficient condition that $\mathrm{cf}(\alpha)>0$ is that there exists an $\aleph_\alpha$-homogeneous set such that every one of its subsets of power $\aleph_\alpha$ contains a well-ordered subset of power $\aleph_\alpha$. *S. Ginsburg.*

Popruženko, J. Sur l'égalité $2^{\aleph_\lambda}=\aleph_{\lambda+1}$. Fund. Math. 43 (1956), 148–155.

Lemma. If $\aleph_0 \leqq l<m$, then for any partially ordered set M of power m, the following propositions (A), (B) are equivalent: (A) (1) Every subset of M of power $\leqq l$ has an upper bound in M, and (2) there exists no cardinal between l and m; (B) (3) M contains a cofinal well-ordered subset of power m, and (4) M contains a subset K of power m such that the set of predecessors in K of each element of M is of power $\leqq l$. Using this lemma, the author proves that the assertion $2^{\aleph_\lambda}=\aleph_{\lambda+1}$ is equivalent to certain statements about transfinite sequences. For any sequence $t=(n_k)$ of positive integers, let $\prod_i t$ denote the subsequence $(n_{2^i(2j-1)})$ $(j=1, 2, \cdots)$ of t. The final theorem states that the proposition $2^{\aleph_0}=\aleph_1$ is equivalent to the following: There exists a family Φ of $2^{\aleph_0}$ sequences of positive integers such that, if s is an arbitrary sequence of positive integers, then the set of all t in Φ for which $\prod_i t \neq \prod_i s$ $(i=1, 2, \cdots)$ is at most denumerable.

L. Gillman (Lafayette, Ind.).

Fodor, G. Eine Bemerkung zur Theorie der regressiven Funktionen. Acta Sci. Math. Szeged 17 (1956), 139–142.

Let λ be a limit ordinal, let M be a stationary subset of $W(\lambda)$ (i.e., M meets every closed cofinal set in $W(\lambda)$), and let ϕ be a regressive function defined on M (i.e., $\phi(\xi)<\xi$ for $\xi>0$, and $\phi(0)=0$ in case $0 \in M$). Theorem 1. If $\mathrm{cf}(\lambda)>0$, there exists an ordinal $\alpha<\lambda$, and a stationary subset N of M, such that $\phi(\beta) \leqq \alpha$ for all $\beta \in N$. Theorem 2. If λ is regular and $>\omega$, there exists a stationary subset N of M on which ϕ is constant. [For background, see Fodor, same Acta 16 (1955), 204–206; MR 17, 831.]

L. Gillman (Lafayette, Ind.).

Eyraud, Henri. Le théorème de l'ordinal limite. Ann. Univ. Lyon. Sect. A. (3) 18 (1955), 5–14.

Il est bien connu que les ordinaux de la troisième classe de Cantor peuvent être représentés par des fonctionnelles arithmétiques de plus en plus croissantes. Il est assez remarquable de signaler que, si une fonctionnelle F est majorante totale d'une fonctionnelle G, il existe néanmoins une infinité non dénombrable de tronçons de G qui ont même hauteur que les tronçons correspondants de F. Il est fait application de ce théorème à la détermination d'une suite transfinie de $\aleph_2$ fonctionnelles, dont chacune soit diviseur asymptotique de chacune des précédentes. (Résumé de l'auteur.) *L. Gillman* (Lafayette, Ind.).

Bachmann, Heinz. Stationen im Transfiniten. Z. Math. Logik Grundlagen Math. 2 (1956), 107-116.

A discourse on inaccessible numbers of various kinds. Further details are given in the author's recent book [Transfinite Zahlen, Springer, Berlin, 1955; MR 17, 134].

L. Gillman (Lafayette, Ind.).

Bergmann, Gustav. The representations of S5. J. Symb. Logic 21 (1956), 257–260.

By McKinsey's method [same J. 6 (1941), 117–134; MR 3, 290] it can be shown that a matrix

$$\Gamma=\langle K, D, \cup, \cap, -, C\rangle$$

is a representation of S5, C corresponding to $\Diamond$, if and only if it is a closure algebra and (α) for every $x \in K$, $-C-Cx=Cx$, (β) D is an additive ideal of K such that, for

every $x \in D$, $Cx=V$. It is shown here that (α) is equivalent to (α''): If $x \cap Cy = \Lambda$, then $Cx \cap Cy = \Lambda$. (α) is also equivalent to $x \cap \neg x = \Lambda$, where $\neg x = C - x$. Consequently, the Brouwerian algebra of the closed elements of a closure algebra Γ is Boolean if and only if Γ satisfies (α''). It also follows that a representation of S2 is also one of S5 if and only if it satisfies (α''). *A. Heyting* (Amsterdam).

Anderson, Alan Ross. Independent axiom schemata for S5. J. Symb. Logic **21** (1956), 255-256.

L. Simons [same J. **18** (1953), 309–316; MR **15**, 493] gave an independent system of axioms for S3, with detachment for material implication as the only rule. The author shows that by addition of one axiom S an independent system of axioms for S5 results. S is

$$\vdash[(\sim\Diamond\sim\alpha \mathrel{-\!\!3} \Diamond\alpha) \mathrel{-\!\!3} (\alpha \mathrel{-\!\!3} \sim\Diamond\sim\Diamond\alpha)].$$

A. Heyting (Amsterdam).

★ **Iablonskii, S. B. On the functional completeness in three-valued calculus.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1957. 5 pp.

Translated from Dokl. Akad. Nauk SSSR (N.S.) **95** (1954), 1153–1155. The original Russian article was reviewed in MR **15**, 925.

Meschkowski, Herbert. Rekursive reelle Zahlen. Math. Z. **66** (1956), 189–202.

The author contrast his 'thoroughly constructive' approach to the theory of recursive real numbers with the 'semi classical' approach of Rice [Proc. Amer. Math. Soc. **5** (1954), 784–791; MR **16**, 104] and others: the latter 'presupposes the familiar concept of real number' while he, following Goodstein [e.g. Acta Math. **92** (1954), 171–190; MR **16**, 783] 'starts with recursive sequences (of rationals which converge to recursive real numbers) and their properties'. {This distinction is meaningless because the basic operations on (recursive) real numbers and their magnitude relations can be defined in terms of approximating sequences; more precisely, the definition can be given, e.g. in classical number theory Z of Hilbert-Bernays. In particular, if R_1' is the class of (Gödel numbers of recursively convergent sequences of rationals which converge to) recursive real numbers and R_2' is the class of (Gödel numbers of recursive sequences $a(n)$, $0 \leqq a(n) \leqq 9$, which represent) recursive decimals, R_1' and R_2' can be defined in Z and $R_1' \equiv R_2'$ can be formally proved in Z contrary to the author's assertion in section III: all he shows is that there is an element a of R_1' such that for any given d of R_2' the formula $a \equiv d$ is not provable in Z, where $a \equiv d$ means that $a-d$ is a null sequence. The real difference between the two approaches lies in the interpretation of the logical connectives: in the semi-classical approach a classical proof of $R_1' \equiv R_2'$ is required, in a thoroughly constructive approach one requires (i) a uniformly recursive functional, or at least a partial recursive function $\nu(n)$ such that if $n \in R_1'$ then $\nu(n) \in R_2'$ and n and $\nu(n)$ represent the same real number, (ii) possibly even a 'constructive' proof of (i). (This would establish $R_1' \subset R_2'$; $R_2' \subset R_1'$ is trivial.) There is no such ν.} Among the author's positive results there are some straightforward theorems on rapidly convergent sequences with irrational limits r, which provide a decision method for $r \geqq p/q$ (p and q integers). These results are badly formulated because there is a recursive decision method for $r \geqq p/q$ for every recursive irrational r; for, if r is irrational at all and recursive there is a recursive $F(q)$ such that

$$|r-p/q| > F^{-1}(q);$$

also since r is approximated by a recursively convergent sequence r_n there is a recursive $N(q)$ such that for n, $m \geqq N(q)$, $|r_n - r_m| < \frac{1}{2}F^{-1}(q)$, hence, for $n \geqq N(q)$,

$$|r_n - p/q| > \tfrac{1}{2}F^{-1}(q),$$

also all such r_n will be on the same side of p/q. (The result is expressed in terms of the approximating sequences.) The point of the author's results is simply that the modulus of convergence is very transparent for the class of sequences which he considers, and so he expresses $N(q)$ as a simple uniformly recursive functional of the function F. *G. Kreisel* (Princeton, N.J.).

Spector, Clifford. On degrees of recursive unsolvability. Ann. of Math. (2) **64** (1956), 581-592.

This paper reports certain results of the author's doctoral dissertation; it answers in particular some questions raised in Kleene and Post, Ann. of Math. (2) **59** (1954), 379–407 [MR **15**, 772]. The first of the three sections of the paper deals with questions concerning join and jump operations defined on degrees of recursive unsolvability which were raised at the end of the first section of the cited paper. The middle section describes sets of degrees which fail to have either l.u.b. or g.l.b. The main result of the paper occupies the third section. It is shown here that for every degree **b** three is a degree $\mathbf{c} < \mathbf{a}''$ such that there is no degree **b** with $\mathbf{a} < \mathbf{b} < \mathbf{c}$. This result answers the question raised at the end of the second section of the cited paper. It remains an open question whether the degree **c** can be so chosen that for any **b** with $\mathbf{b} < \mathbf{c}$ we have $\mathbf{b} \leqq \mathbf{a}$. *E. J. Cogan* (Hanover, N.H.)

Grzegorczyk, A. Some proofs of undecidability of arithmetic. Fund. Math. **43** (1956), 166–177.

The author presents a proof of the undecidability of arithmetic, proved originally by Gödel [Monatsh. Math. Phys. **38** (1931), 173–198]. The proof, less constructive than that of Rosser [J. Symb. Logic **1** (1936), 87–91], is based on the following lemma: If a non-computable set X is represented in each consistent and recursively enumerable extension of a theory T, then T is essentially undecidable. A theory T is essentially undecidable if it is consistent and if each consistent extension of T is undecidable. The author furnishes results concerning the form of undecidable sentences of a general theory T. In the latter half of the paper these results are applied to the theory Ar of arithmetic. It is shown that both a diagonal set and a simple set may be represented in Ar. Attempts to repeat the proof of representability of a simple set in theories narrower than Ar have failed. Erratum, p. 171, line 11, for $R(n, x)$ read $R(n, k)$. *E. J. Cogan* (Hanover, N.H.).

See also: Jaffard, p. 555; McCluskey, p. 624.

ALGEBRA

★ **Chevalley, Claude. Fundamental concepts of algebra.** Academic Press Inc., New York, 1956. viii+241 pp. $6.80.

A generation of algebraists grew up for whom "modern algebra" meant Van der Waerden's book, or possibly one of several similar later texts. Time has passed and (happily) mathematics has not stood still. In particular algebraic topology has exhibited an insatiable appetite for algebraic gadgets. In response, modern algebra has changed.

What distinguishes the new modern algebra from the old? The latter emphasized groups, rings, and homomorphisms as the basic concepts. Modules, more or less sitting astride groups and rings, were prominent, though perhaps not sufficiently prominent. But at least two things, now clearly of central importance, were completely missing: the tensor product of modules, and the generalization of every object to a graded object.

Chevalley's book is timely and it will be widely studied; the meaty exercises will invite a diligent reader to educate himself some more. Teachers may find it "futile to disguise the austerity" (last sentence of the preface). It goes without saying that large sections are similar to Bourbaki's "Algèbre multilinéaire" [Actualités Sci. Ind., no. 1044, Hermann, Paris, 1948; MR 10, 231].

A rough survey of the contents is as follows. I. Monoids (semigroups with unit). II. Groups (including free groups). III. Rings and modules (including tensor products, duality graded modules). IV. Algebras (a skeleton introduction). V. Associative algebras. This final chapter occupies nearly half the book. A selection from the subheadings will indicate the territory traversed: derivations, Grassmann algebras, determinant, trace, Pfaffian, symmetric algebras. At the end of this long road the final object unveiled to the reader is — a polynomial algebra! *I. Kaplansky.*

Combinatorial Analysis

See: Hall, p. 560; Gleason, p. 593; Bottema, p. 629.

Elementary Algebra

Blanuša, Danilo. Quelques identités algébriques concernant les moyennes arithmétique et géométrique. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II. **11** (1956), 17–22. (Serbo-Croatian summary)

A proof that the arithmetic mean is not less than the geometric mean. It is based on the following lemma, which is proved by induction.

If $d_i>0$, $i=0, 1, \cdots, n-1$, then

$$\Big(\sum_0^{n-1}(n-i)d_i\Big)^n \geqq n^n \prod_0^{n-1}\sum_0^i d_j.$$

R. L. Jeffery (Kingston, Ont.).

Devidé, Vladimir. Ein Vergleich des arithmetischen und geometrischen Mittels. Glasnik Mat.-Fiz. Astr. Društvo Mat. Fiz. Hrvatske. Ser. II. **11** (1956), 23–24. (Serbo-Croatian summary)

If S is the arithmetic mean, G the geometric mean of n numbers $x_1x_2\cdots x_n$, then it is known that $S-G>0$. For $n=2$

$$S-G=\frac{x_1+x_2}{2}-\sqrt{(x_1x_2)}=\frac{(\sqrt{x_1}-\sqrt{x_2})^2}{2}>0,$$

$$S^2-G^2=\Big(\frac{x_1+x_2}{2}\Big)^2-x_1x_2=\Big(\frac{x_1-x_2}{2}\Big)^2>0.$$

The usual proof of the corresponding result for $n>2$ is by induction [Hardy, Littlewood and Polya, Inequalities, Cambridge, 1934]. The author obtains simple proofs by using algebraic identities which are generalizations of an identity of Hurwitz [J. Reine Angew. Math. **108** (1891), 266–268]. *R. L. Jeffery* (Kingston, Ont.).

Palamà, Giuseppe. Polinomi interi in x di grado n dispari che assumono n volte ciascuno dei $2m$ valori $\pm N_1, \cdots, \pm N_m$. Boll. Un. Mat. Ital. (3) **11** (1956), 368–370.

Generalizing a theorem by Dorwart [Duke Math. J. **1** (1935), 70–73] a polynomial in x, of odd degree n, with integral coefficients is constructed, that gives n times all the $2m$ values $\pm N_i$ ($i=1, 2, \cdots, m$) for $n\cdot 2m$ integral and different values of x. N_i are suitable natural numbers. A theorem on equal sums of like powers is used.

N. G. W. H. Beeger (Amsterdam).

Linear Algebra

★ **Artin, E. Geometric algebra.** Interscience Publishers, Inc., New York-London, 1957. x+214 pp. $6.00.

Depuis 20 ans, tous les mathématiciens qui se sont occupés d'algèbre linéaire savent combien le langage géométrique „intrinsèque" permet de clarifier et de simplifier cette théorie. Néammoins, il ne se passe guère d'année sans qu'il ne se publie un ou deux „textbooks" de „calcul matriciel", le plus souvent d'une affligeante médiocrité, mornes compilations de recettes de calcul sans motivation ni intérêt apparents. Il est réconfortant de voir un mathématicien, aussi universellement respecté que l'est Artin, prendre la peine de montrer par l'exemple comment on peut, de la façon la plus attrayante possible, conduire l'étudiant, à partir de notions d'algèbre tout à fait rudimentaires, jusqu'aux problèmes les plus difficiles de la théorie des groupes classiques. L'expérience est une éclatante réussite; on ne sait ce qu'il faut le plus admirer, de la parfaite lucidité du style, qui sait ne jamais être pesant ni pédant, de la maîtrise dans l'agencement des lemmes et théorèmes, ou de la perpétuelle ingéniosité qui se manifeste dans chaque démonstration; et le spécialiste pourra constater que, sauf peut-être dans les parties les plus classiques du volume, il n'y a pour ainsi dire pas une démonstration où l'auteur n'ait apporté sa contribution personnelle et ne soit parvenu à améliorer en simplicité et en élégance (parfois de façon très appréciable) les exposés existants.

Le chapitre I développe les notions de base: groupes, corps, espaces vectoriels, dualité. Mais déjà on y trouve maint détail sortant de l'ordinaire, comme la démonstration de Witt du théorème de Wedderburn sur les corps finis, une très jolie démonstration du théorème de Hua sur les automorphismes d'un corps, l'exemple du corps ordonné non commutatif de Hilbert, et une intéressante discussion des corps ordonnés, où l'auteur montre qu'un

affaiblissement apparent des axiomes conduit en fait à la théorie usuelle.

Au chap. II est exposée la théorie traditionnelle de la géométrie projective axiomatique: introduction des coordonnées, théorèmes de Desargues et de Pappus et „théorème fondamental de la géométrie projective"; mais, contrairement à la plupart de ses prédécesseurs, l'auteur évite les sentiers battus, et en fait la plus grande partie du chapitre traite de la géométrie affine, le langage „projectif" n'étant introduit que tout à la fin — ce qui, de l'avis du rapporteur, a l'avantage de clarifier considérablement l'exposé.

Les formes bilinéaires et quadratiques forment l'objet du chap. III, intitulé „Géométrie symplectique et géométrie orthogonale". Faute de place, l'auteur laisse de côté les formes hermitiennes et les formes quadratiques sur un corps de caractéristique 2. Toute la théorie est bien entendu développée sous forme géométrique, et la géométrie symplectique et la géométrie orthogonale sont menées de front aussi longtemps que cela est possible (ce qui inclut, en particulier, les propriétés des sous-espaces isotropes et le théorème de Witt); il faut en particulier signaler le paragraphe consacré aux géométries sur un corps fini, véritable morceau de bravoure où l'auteur arrive à réduire les calculs pratiquement à rien. On notera aussi que l'élimination de la géométrie hermitienne et de la caractéristique 2 permet à certains endroits (notamment dans le théorème de Witt) de notables simplifications, mais on peut parfois se demander si, en ne présentant pas les démonstrations valables en toute généralité, l'auteur ne risque pas de donner à l'étudiant une idée un peu déformée de la situation.

Le chap. IV est, à la connaissance du rapporteur, le premier exposé didactique complet de la théorie du groupe linéaire sur un corps non nécessairement commutatif, y compris la théorie des déterminants; dans les grandes lignes, l'auteur suit les travaux de Hua et du rapporteur sur ces questions, non sans y apporter, ici encore, des simplifications appréciables, en particulier dans le cas épineux de la dimension 2.

Le dernier chapitre enfin aborde la partie la plus difficile du sujet, la structure des groupes orthogonaux; auparavant l'auteur dispose rapidement des groupes symplectiques, et prend soin, avant de plonger dans le vif du sujet, d'insérer un bref paragraphe sur le cas „euclidien" classique, pour faire voir clairement comment les propriétés particulières du corps réel masquent ici les difficultés de la question; ce qu'il met en lumière en faisant aussitôt suivre ce paragraphe d'un traitement très original et suggestif du cas en quelque sorte „opposé", celui des géométries qu'il appelle „elliptiques" sur un corps valué, où le groupe orthogonal admet une infinité de sous-groupes distingués. Viennent ensuite l'algèbre de Clifford et la norme spinorielle; la limitation imposée par le caractère élémentaire des connaissances de base supposées chez le lecteur empêche l'auteur de donner ici un exposé vraiment moderne du sujet, mettant en particulier en relief le caractère „universel" de l'algèbre de Clifford; du moins le traitement élémentaire de la théorie est-il complet, sans aucune „tricherie" sur les détails ennuyeux, comme la vérification de l'associativité. Enfin le chapitre se termine par les théorèmes fondamentaux sur la structure du groupe orthogonal (en se bornant au cas non anisotrope pour les dimensions $\geqq 5$); ici les démonstrations sont nouvelles et sont de loin les plus simples de celles qui ont été proposées jusqu'ici.

Il serait injuste de ne pas rappeler que, dans ces dernières années, ont déjà été publiés plusieurs ouvrages de valeur sur l'algèbre linéaire „géométrique"; mais si l'on peut parler de „définitif" en quoi que ce soit, il semble au rapporteur que le livre d'Artin mérite ce qualificatif plus que tout autre. Connaissant le conservatisme foncier de l'enseignement, ce serait faire preuve d'un optimisme un peu naïf que d'imaginer toutefois que ce volume va du jour au lendemain bouleverser les cours poussiéreux de „géométrie projective" ou d',,algèbre des matrices" que doivent subir la plupart des étudiants d'aujourd'hui; mais s'il y a sans doute encore de beaux jours pour les faiseurs de manuels de calcul matriciel, où la routine le disputera à l'incompétence, on peut s'en consoler en pensant que, d'ici une ou deux générations, l'enseignement aura vraisemblablement avancé assez loin pour que le monde mathématique tout entier, et non seulement une poignée de spécialistes, soit mis en état d'apprécier l'ouvrage d'Artin et de le mettre à la place qui lui revient, à côté des célèbres „Grundlagen der Geometrie" de Hilbert. *J. Dieudonné* (Evanston, Ill.).

Carlitz, L.; and Hodges, John H. Distribution of matrices in a finite field. Pacific J. Math. **6** (1956), 225–230.

Some enumerative problems concerning square matrices with elements in a finite field are considered. The following are determined: (1) the number of non-derogatory matrices of order m, (2) the number of equivalence classes of similar matrices of order m having a specified characteristic polynomial, (3) the total number of equivalence classes of similar matrices of order m, and (4) the number of equivalence classes of similar matrices of order m with minimum polynomial of degree $\leqq m$. Also, a partial result is obtained for the problem of determining the number of admissible minimum polynomials of fixed degree for matrices of order m.

The method employed is to reduce matrix problems to problems concerning polynomials. *G. L. Walker.*

Pták, Vlastimil. Eine Bemerkung zur Jordanschen Normalform von Matrizen. Acta Sci. Math. Szeged **17** (1956), 190–194.

Let A be a linear transformation of the finite dimensional vector space X. The author shows how the theory of duality [cf. N. Bourbaki, Algèbre linéaire, Actualités Sci. Ind., no. 1032, Hermann, Paris, 1947; MR **9**, 406] can be exploited to give a short and illuminating geometric proof of the classical theorem: there exists a basis of X relative to which the matrix of A has the Jordan normal form. *B. N. Moyls* (Vancouver, B.C.).

Paasche, Ivan. Bemerkung zu einem Desideratum von Perron. Math. Z. **66** (1956), 117–120.

Perron [Math. Z. **64** (1955), 103–114; MR **17**, 1175] indirectly established the equality of two determinants built out of sums of powers, and inquired whether a more direct proof could be found. The author exhibits an explicit matrix of determinant one that accomplished the passage from one to the other. *I. Kaplansky.*

See also: MacLane, p. 558; Ballantine, p. 561; van der Waerden, p. 562; Reiner and Swift, p. 565; Frölicher and Nijenhuis, p. 569; Bellman, p. 576; Chang, p. 626.

Polynomials

Parodi, Maurice. Sur la localisation des zéros des polynômes. C. R. Acad. Sci. Paris **243** (1956), 1093–1096.

By application of the results of his previous paper

[same C. R. **242** (1956) 2617–2618; MR **18**, 4] the author establishes that, if $|a_1|>1+nA$ and $A>(n-1)^{-1}$, then the polynomial $f(z)=z^n+a_1z^{n-1}+\cdots+a_{n-1}z+a_n$ has $n-1$ zeros in the unit circle and exactly one zero in the circle

$$|z+a_1|\leq\tfrac{1}{2}\{|a_1|-(|a_1|^2-4nA)^{\frac{1}{2}}\}.$$

M. Marden (Milwaukee, Wis.).

Zamansky, Marc. Algèbre des polynômes. Enseignement Math. (2) **2** (1956), 293-306.

A pedagogical presentation in which analysis is carefully banned.

See also: Carlitz and Hodges, p. 554.

Partial Order Structures

★**Dantzig, G. B.; and Hoffman, A. J. Dilworth's theorem on partially ordered sets.** Linear inequalities and related systems, pp. 207–214. Annals of Mathematics Studies, no. 38. Princeton University Press, Princeton, N. J., 1956. $5.00.

The authors apply linear programming methods to give a new proof of the finite case of a theorem of the reviewer on decompositions of partially ordered sets [Ann. of Math. (2) **51** (1950), 161–166; MR **11**, 309]. This theorem asserts that in a partially ordered set P the smallest number of disjoint chains whose set union is P is the largest number of mutually unrelated elements in P. The linear programming problem is defined as follows: Let $a_1, \cdots, a_n$ be the elements of P. Define $c_{ij}=0$ if $a_i<a_j$ and $c_{ij}=-\infty$ if $a_i\not<a_j$ for $i, j=1, \cdots, n$. Finally, set $c_{00}=1$, $c_{0i}=c_{i0}=0$ for $i=1, \cdots, n$. Then the problem is to maximize $\sum_{ij=0}^{n} c_{ij}x_{ij}$ under the restrictions: 1) $x_{ij}\geq 0$, 2) $\sum_{i=0}^{n} x_{0i}=\sum_{i=0}^{n} x_{i0}=n$ for $i=0, \cdots, n$, 3) $\sum_{i=0}^{n} x_{ij}=\sum_{i=0}^{n} x_{ji}=1$ for $j=1, \cdots, n$. The authors show that the decomposition theorem for partially ordered sets follows from the duality theorem when applied to the linear programming problem. *R. P. Dilworth.*

Petropavlovskaya, R. V. Associative systems that are lattice-isomorphic to a given group. I. Vestnik Leningrad. Univ. **11** (1956), no. 13, 5–26. (Russian)

Two associative systems are said to be lattice isomorphic if the lattices of their subsystems are isomorphic. Typical results of this paper are concerned with: A necessary and sufficient condition that the subsystems of an associative system are linearly ordered by inclusion; the associative systems lattice isomorphic to a cyclic group of prime power order; the theorem that the necessary and sufficient condition for the subgroup lattice of a group be a non-trivial direct sum of sublattices is that G be periodic and decomposes into a direct product of proper subgroups of relatively prime orders. *W. T. van Est.*

Szász, G. Die Translationen der Halbverbände. Acta Sci. Math. Szeged **17** (1956), 165-169.

A translation of a semigroup S is a function λ of S into itself satisfying $\lambda(xy)=\lambda(x)\cdot y$. Assume henceforth that S is a semilattice (=idempotent commutative semigroup). It is shown that a translation λ of S is an idempotent endomorphism with $\lambda(S)$ an ideal, and conversely if φ is an idempotent endomorphism with $\varphi(S)$ an ideal, then φ is a translation. It is also shown that a translation of S is a closure operation [Birkhoff, Lattice theory, Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948; MR **10**, 673] in the ordering $x\leq y\leftrightarrow xy=y$. *R. J. Koch.*

Jaffard, Paul. Un problème sur les ensembles lié à théorie de la croissance. Bull. Sci. Math. (2) **80** (1956), 100–108.

The author asks whether there exists a free ultrafilter with the countable intersection property, and shows that none can exist on any set of power $\leq c$. However, the following far stronger result has been known for a quarter of a century: there exists no such ultrafilter on any set whose cardinal is smaller than the first strongly inaccessible cardinal. (This is the celebrated result of Ulam and Tarski, stated originally in the equivalent terms of a free, two-valued (countably additive) measure.) The author goes on to prove a number of theorems about real-valued functions, every one of which (with a minor exception) was proved by Hewitt, in 1948, in considerably greater generality [Trans. Amer. Math. Soc. **64** (1948), 45–99, Th. 36, 41 and 50; MR **10**, 126; see also Gillman, Henriksen and Jerison, Proc. Amer. Math. Soc. **5** (1954), 447–455, Th. 2; MR **16**, 607]. *L. Gillman.*

Sikorski, R.; and Traczyk, T. On some Boolean algebras. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 489–492.

The authors investigate σ-Boolean algebras A, which are isomorphic to a Boolean algebra of the form $\mathfrak{S}(\mathfrak{X})/I$, where $\mathfrak{S}(\mathfrak{X})$ is the Boolean algebra of all the subsets of some set $\mathfrak{X}$ and I is a σ-ideal of $\mathfrak{S}(\mathfrak{X})$. Let the topological space $\mathfrak{Y}$ be an absolute B-retract, let $\mathfrak{C}_\mathfrak{m}$ denote the generalised Cantor discontinuum ($\mathfrak{m}$ a cardinal), let $\mathfrak{A}(\mathfrak{Y})$ and $\mathfrak{A}(\mathfrak{C}_\mathfrak{m})$ denote the least σ-fields containing all sets which are both open and closed in $\mathfrak{Y}$ and in $\mathfrak{C}_\mathfrak{m}$ respectively. Using results of previous papers of the first author [Fund. Math. **36** (1949), 7–22; **38** (1951), 53–54; Colloq. Math. **2** (1951), 202–211; MR **11**, 166; **14**, 347, 71] the authors prove that every σ-homomorphism of $\mathfrak{A}(\mathfrak{Y})$ into $\mathfrak{A}$ can be extended to a σ-homomorphism of $\mathfrak{S}(\mathfrak{Y})$ into A and that every σ-homomorphism of $\mathfrak{A}(\mathfrak{C}_\mathfrak{m})$ into A can be extended to a σ-homomorphism of $\mathfrak{S}(\mathfrak{C}_\mathfrak{m})$ into$A$, for every $\mathfrak{m}$. Each of these conditions is also sufficient in order that a σ-Boolean algebra A is of the above type. {There is a misprint in (i) (4): the second "into" should be "onto".} If $\mathfrak{Y}$ is some complete separable metric space and A a Boolean algebra isomorphic to $\mathfrak{S}(\mathfrak{X})/\mathfrak{J}$ for some set $\mathfrak{X}$ (where I is a σ-ideal), then every σ-homomorphism of $\mathfrak{B}(\mathfrak{Y})$ into A can be extended to a σ-homomorphism of $\mathfrak{S}(\mathfrak{Y})$ into A, where $\mathfrak{B}(\mathfrak{Y})$ is the σ-field of all Borel subsets of Y. The converse need not be true: If A is the least $2^{\aleph}$-additive field of subsets of $\mathfrak{C}_\mathfrak{m}$, containing $\mathfrak{A}(\mathfrak{C}_\mathfrak{m})$, where $\mathfrak{m}=2^{2^{\aleph}}$, then A satisfies the preceding condition, but A is not a σ-Boolean algebra of the above type.

Ph. Dwinger (Lafayette, Ind.).

★**Povarov, G. N. On the functional separability of Boolean functions.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 5 pp.

Translation from Dokl. Akad. Nauk SSSR (N.S.). **94** (1954), 801–803. The original Russian article was reviewed in MR **16**, 107.

Rings, Fields, Algebras

Ramanathan, K. G. Quadratic forms over involutorial division algebras. J. Indian Math. Soc. (N.S.) **20** (1956), 227–257.

Let K be a field which is either an algebraic number

field or an algebraic function field of one variable with finite constant field, and D be an involutorial non-commutative division algebra with K as center. The author considers quadratic forms over D. In the case where the involution of D leaves K fixed it is well known that D is a quaternion algebra over K. The author shows that in the function field case every quadratic form $f(x)$ in n variables is a universal form and is a null form if $n>1$. In the algebraic number field case he shows that the form $f(x)$ represents an element $b\neq 0$ of K if and only if this occurs in every completion of K. Also two nonsingular forms in n variables are equivalent in D if and only if they are equivalent in every completion. The author also makes a study of forms over involutorial algebras in which the involution moves the center and gives some results for skew forms. *A. A. Albert.*

Dwork, B. On the Artin root number. Amer. J. Math. 78 (1956), 444–472.

Let Ω be the algebraic closure of the field Q of rational numbers and let Γ be its Galois group. Let $\mathfrak{X}_{gl}$ be the set of objects (H_1, H_2, χ), where H_1, H_2 are closed subgroups of finite index of Γ with H_2 normal subgroup of H_1 and χ is a character of H_1/H_2. If $X=(H_1, H_2, \chi), X'=(H_1,H_2,\chi')$, we set $X+X'=(H_1, H_2, \chi+\chi')$. Assume that H_3 lies between H_1 and H_2; if $X=(H_3, H_2, \chi)$ and χ' is the character of H_1/H_2 induced by χ, we say that (H_1, H_2, χ') is induced by X; if H_3 is normal in H_1 and $X=(H_1, H_3, \chi)$, then χ determines in a natural manner a character χ' of H_1/H_2, and we say that (H_1, H_2, χ') is lifted from (H_1, H_3, χ); we say that $X=(H_1, H_2, \chi)$ and $X'=(H_1', H_2', \chi')$ are equivalent if there is an element s of Γ such that $H_i'=sH_is^{-1}$ $(i=1, 2)$ and such that χ' corresponds to χ in the resulting isomorphism of H_1/H_2 with H_1'/H_2'. A mapping F of $\mathfrak{X}_{gl}$ into a multiplicative group will be called a D-function if it has the following properties: $F(X+X')=F(X)F(X')$ if $X+X'$ is defined; $F(X)=F(X')$ if either X' is equivalent to X or induced by X or lifted from X. We have entirely similar notions if we start with some p-adic completion Q_p of Q instead of Q itself; we then define a set $\mathfrak{X}_p$ of triplets (H_1, H_2, χ) and the notion of D-function on $\mathfrak{X}_p$. For every closed subgroup H of Γ (in the global case) let $k(H)$ be the fixed field of H; let $X=(H_1, H_2, \chi)\in\mathfrak{X}_{gl}$ and let $\mathfrak{p}$ be a prime divisor of k_{H_1}; by restricting χ to the decomposition group of some place of k_{H_2} above $\mathfrak{p}$, we associate to X a class of equivalent elements $X_\mathfrak{p}$ of $\mathfrak{X}_p$ (where p is the rational prime above $\mathfrak{p}$).

To every $X=(H_1, H_2, \chi)$, there is associated a number $W(X)$, the Artin root number, which occurs in the functional equation of the Artin L-series relative to χ. It is known that the function $X\rightarrow W(X)$ is a D-function on $\mathfrak{X}_{gl}$. On the other hand, it is known that, if one limits oneself to linear characters X (i.e. $X=(H_1, H_2, \chi)$ with χ of degree 1), then $W(X)$ may be represented as a product of numbers $W_\mathfrak{p}(X)$ associated to the various places of the fixed field of H_1. The author investigates the problem of finding a similar representation for $W(X)$ in the general case. He proves the following: let Δ be the factor group of the multiplicative group of complex numbers $\neq 0$ by the group composed of ± 1; let $W^*(X)$ be the class of $W(X)$ in Δ. Then one can define for each prime divisor p of Q a D-function F_p on $\mathfrak{X}_p$ with values in Δ such that $W^*(X)=\prod_\mathfrak{p} F_p(X_\mathfrak{p})$, the product being extended over all prime divisors $\mathfrak{p}$ of the fixed field associated to X and p denoting the prime divisor of Q above $\mathfrak{p}$. Moreover, enough can be proved about the local functions F_p to obtain a proof of the Hasse conjecture which says that $(Nf(X))^{\frac{1}{2}}W(X)$ is an algebraic integer (where $Nf(X)$ is the absolute norm of the conductor of X) and to give a sharp statement about the field generated by the number $W(X)$. As for the methods, the essential problem is to extend to the whole of $\mathfrak{X}_p$ the functions F_p which were already known for the linear characters. A large portion of this extension problem is solved in a purely group-theoretic manner; however, these methods do not suffice entirely, and the author has to complete them by making use of the fact that the function $W(X)=\prod_\mathfrak{p} F_p(X)$ is already extended to the whole of $\mathfrak{X}_{gl}$. *C. Chevalley* (Paris).

Abhyankar, Shreeram. On the valuations centered in a local domain. Amer. J. Math. 78 (1956), 321–348.

If n is the finite transcendence degree of the field K over its prime subfield P, the absolute dimension of K is defined to be n or $n+1$, according as the characteristic p of P may be $p\neq 0$ or $p=0$ respectively. If K is the field of rational functions over a variety V and if n is the absolute dimension of K over P, it is said that V is an absolute n-dimensional variety. The author proves that the singularities of a curve lying on a non-singular absolute surface can be resolved by quadratic transformations applied to the surface. The author denotes by (R, M) a local domain R with M as maximal ideal. If $x_1, \cdots, x_s$ is a minimal basis of M, and v a valuation of the quotient field K of R, having center M in R, if $v(x_1)\leq v(x_i)$, $i=1, \cdots, s$, if $A=R[x_2/x_1, \cdots, x_s/x_1]$, $P=A\cap M_v$, $S=A_P$ and $N=PS$, then it is said that (S, N) is the first quadratic transform of R along v. One says that S is a quadratic transform of R if S is the nth quadratic transform of R along some valuation v of K with center on M. Let (R, M) be a two dimensional regular local domain; the author proves: i) If f is a non zero element of R, there exist a quadratic transform (R^*, M^*) of (R, M) and a basis x^*, y^* of M^* such that $f=x^{*a}y^{*b}d$, where a and b are non-negative integers and d is a unit in R^*. ii) If (R', M') is another regular two dimensional local domain, with a common quotient field K with R, such that $R\subset R'$; $M'\cap R=M$; then R' is a quadratic transform of R.

In the first part of the paper are obtained some generalizations of properties of valuations of a field K of finite transcendence degree over its prime field which give relations among the rational rank, the rank, the dimension of K and the dimension of the valuation relative to some local domain. *P. Abellanas* (Madrid).

★ **Chatelet, Albert. Arithmétique et algèbre modernes. Tome Second. Anneaux et corps - Calcul algébrique, idéaux et divisibilité.** Presses Universitaires de France, Paris, 1956. vii+pp. 277–728. 1800 francs.

This second volume [for v. 1, 1954, see MR 15, 773] is concerned mainly with commutative rings with unity. The first chapter is on the elementary theory of rings, modules, and algebras. In the second chapter, polynomial rings, bilinear operations, and matrices are discussed. The final chapter is on the structure theory of commutative rings. It has a discussion of radicals, Artin rings, Noetherean rings, prime and semi-prime ideals, and the decomposition of the ideals of a ring. *R. E. Johnson.*

Bryant, S. J.; and Zemmer, J. L. A note on completely primary rings. Proc. Amer. Math. Soc. 8 (1957), 140–141.

A completely primary ring A is a commutative ring

with identity whose radical (i.e., the ideal of nilpotent elements) is maximal. E. Snapper [Ann. of Math. (2) **53** (1951), 207–234; MR **12**, 584] showed that if A has characteristic 0, then A contains a field F isomorphic to A/N. The authors present an alternate proof of Snapper's result, and show moreover that F can be chosen so that A is additively the direct sum of F and N. *M. Henriksen.*

Albert, A. A. A property of ordered rings. Proc. Amer. Math. Soc. **8** (1957), 128–129.

The author shows that if R is an ordered regular ring in the sense of O. Ore (i.e., every pair of nonzero elements has a common nonzero right multiple), then the quotient ring of R can be ordered in the natural way, and uses this to obtain a simple proof of a theorem of Wagner [Math. Ann. **113** (1936), 528–567]; namely, every ordered ring satisfying a nontrivial polynomial identity is commutative. *M. Henriksen* (Princeton, N.J.).

Baxter, Willard E. Lie simplicity of a special class of associative rings. Proc. Amer. Math. Soc. **7** (1956), 855–863.

The author extends the characterization of the Lie ideals of the commutator subgroup of a simple associative ring to cover the cases of characteristic 2 and 3, and thereby completing the cases left open by the reviewer [Duke Math. J. (1955), 471–476; MR **17**, 577]. The theorem he proves can now be made to read: If A is a simple ring and if U is a proper Lie ideal of $[A, A]$, then U is contained in the center of A except in the case where A is a 4-dimensional algebra over a field of characteristic 2. He then uses these results to characterize the invariant subspaces of central simple finite dimensional algebras and from this obtains a simple proof of a result due to Hattori on invariant subrings in matrix rings.

{Reviewer's note: Amitsur, in the paper reviewed above, using an argument similar to Baxter's is able to characterize the invariant subrings of simple rings with an idempotent $e\neq 1$, $e\neq 0$ as central (except for certain exceptions) or the entire ring.} Note: (I) on page 860 should read: $ut-tu-utu \in T$ (not $\notin T$).

I. N. Herstein (New Haven, Conn.).

Amitsur, S. A. Invariant submodules of simple rings. Proc. Amer. Math. Soc. **7** (1956), 987–989.

The reviewer had conjectured that if a subring of a simple ring was invariant under all automorphisms of the ring then the subring was either the full ring or was contained in the center of the ring. In this paper an example is given which shows that this conjecture, in its full generality, is false. The main point of the paper is to prove: if a simple ring A has an idempotent different from 0 and 1 and if the ring is not 4-dimensional over its center, a field of characteristic 2, then any invariant subspaces (under inner automoropisms) are either contained in the center of A or contain $[A, A]$; in particular the only invariant subalgebras are the center of A and A itself. The proof is to show that the invariant subspace is a Lie ideal relative to $[A, A]$ and to invoke the structure of such Lie ideals as determined by the reviewer (and rederived by the present author). [See Herstein, Duke Math. J. **22** (1955), 471–476; MR **17**, 577; Hattori, Jap. J. Math. **21** (1951), 121–129; MR **14**, 529; see also the paper reviewed above.] *I. N. Herstein* (New Haven, Conn.).

Herstein, I. N. Conjugates in division rings. Proc. Amer. Math. Soc. **7** (1956), 1021–1022.

In this article a theorem is proved equivalent to the assertion that a non-central element of a division ring D possesses infinitely many conjugates (with respect to the center Z). This result is included in a corollary to a theorem of Hattori [J. Math. Soc. Japan **4** (1952), 205–217; MR **14**, 723] when D has finite rank over Z. The present theorem which is valid for D of infinite rank is proved with the use of the Cartan-Brauer-Hua theorem on invariant subrings of division rings [Brauer, Bull. Amer. Math. Soc. **55** (1949), 619–620; MR **10**, 676]. One notes, however, that the application of this latter theorem may be eliminated entirely in a shortened version of the author's proof: Let N be the cetralizer of $\theta \in D$, and let N', D' be the groups of non-zero elements of N and D, respectively. If θ has finitely many conjugates, then N' has finite index in D' (all of this as in the author's proof). If N' has only finitely many elements, then D' is finite, so that $D=Z$ by the Wedderburn theorem [Trans. Amer. Math. Soc. **6** (1905), 349–352]. Otherwise, N' has infinitely many elements so that $N=D$ (as in the last paragraph of the author's proof when Z is replaced by N, and a denotes an arbitrary element of D). Thus, $\theta \in Z$, completing the proof. The following corollary given in the article under review should be of universal interest: If a polynomial $p(x)$ of degree n (with coefficients in Z) has $n+1$ roots in D, then $p(x)$ has infinitely many roots in D.

C. C. Faith (East Lansing, Mich.).

O'Meara, O. T. Basis structure of modules. Proc. Amer. Math. Soc. **7** (1956), 965–974.

Let R be a Dedekind ring (integral domain in which classical ideal theory holds). Any finitely generated torsion free R-module M is a direct sum of ideals. Let us take the point of view that M is embedded in the quotient field of R. Then more can be said: for two such modules it is possible to choose simultaneous bases. This result fails for three or more modules, and the author is mainly concerned with finding conditions under which it survives. These are first stated for the case where R is a valuation ring, and then the global statement is that the conditions must hold at each prime ideal. *I. Kaplansky.*

★ **Ramanathan, K. G. Units of fixed points in involutorial algebras.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 103–106. Science Council of Japan, Tokyo, 1956.

Let A be a simple algebra of finite rank over the field Γ of rational numbers, with an involutorial anti-automorphism $*$ so that $(\alpha+\beta)^*=\alpha^*+\beta^*$, $(\alpha\beta)^*=\beta^*\alpha^*$, $(\alpha)^{**}=\alpha$. If $\xi^*=\xi$, for ξ in A, ξ is called a fixed point of the involution. Let Q be an order in A relative to the rational integers. If u^{-1} and $u \in Q$, u is a unit of Q, and is said to be a unit of the fixed point ξ if $u^*\xi u=\xi$. The set $\Gamma(\xi)$ of such u's is called the unit group of ξ. The author outlines a proof of Theorem 1: If $\xi=\xi^*$, or $\xi=-\xi^*$, and the norm of ξ is $\neq 0$, then $\Gamma(\xi)$ has a finite set of generators. He states a second theorem asserting the finiteness of the non-euclidean "volume", i.e., the measure, of a fundamental region for $\Gamma(\xi)$, which is representable as a properly discontinuous group of transformations of a certain Riemannian space. [Cf. C. L. Siegel, Ann. of Math. (2) **44** (1943), 674–689; Math. Ann. **124** (1952), 364–387; MR **5**, 228; **16**, 801.] The author states that detailed proofs of the theorems are to appear later. *R. Hull.*

McCoy, Neal H. Annihilators in polynomial rings. Amer. Math. Monthly **64** (1957), 28–29.

If R is an associative ring and A a right ideal in

$R[x_1, \cdots, x_n]$ whose right annihilator B is non-zero, then $B \cap R \neq 0$. In particular, if R is commutative and f is a divisor of zero in $R[x_1, \cdots, x_n]$, then there is a non-zero c in R such that $cf=0$, as the author had shown earlier [McCoy, same Monthly **49** (1942), 286–295; MR **3**, 262].

W. R. Scott (Lawrence, Kan.).

Seidenberg, A. An elimination theory for differential algebra. Univ. California Publ. Math. (N.S.) **3** (1956), 31–65.

Let K be an ordinary differential field of characteristic p; let $F_1, \cdots, F_s, G$ be elements of the differential polynomial ring $K\{U_1, \cdots, U_n\}$. For $p=0$ an algorithm is given for deciding whether the system

$$(1) \qquad F_1 = \cdots = F_s = 0, \; G \neq 0,$$

has a solution in some extension of K. This algorithm is constructive in the strong sense that it employs only the differential field operations (and not factorizations). As an application, a constructive nullstellensatz is proved.

If $p \neq 0$, in defining constructibility it turns out to be important to allow not only the field operations but also extractions of pth roots of pth powers in $K\{U_1, \cdots, U_n\}$. Assuming that the field of constants K_0 of K is K^p, the author again obtains a constructive nullstellensatz, and from this an elimination algorithm for (1). Even if $K_0 \neq K^p$, these results continue to hold (but with loss of constructivity) provided the word "solution" is interpreted to mean "separable solution". He also studies, from the constructive point of view, the question of reducing (1) to certain special kinds of such systems.

When K is a partial differential field, the algorithm and nullstellensatz for $p=0$ are proved through the use of certain "maximal canonically reduced systems", which are too complicated to describe here. Such a system has the property that if it has a solution when considered merely as a set of polynomials in the U's and their derivatives, then it has a solution when considered as a set of differential polynomials; thus for such a system the classical algorithm for polynomial algebra is already an algorithm for differential algebra. As an application, a proof is given of a theorem on the extensibility of specializations over differential fields, previously known only for ordinary differential fields [Ritt, Trans. Amer. Math. Soc. **48** (1940), 542–552, pp. 543–545; MR **2**, 197]. Bounds on the number of steps in the algorithm are also studied. For $p \neq 0$, similar methods give a nullstellensatz and a decision method (but not an algorithm) for (1), valid for all solutions if $K_0 = K^p$, and for separable solutions in any case. *A. Rosenfeld* (New York, N.Y.).

MacLane, Saunders. Slide and torsion products for modules. Univ. e Politec. Torino. Rend. Sem. Mat. **15** (1955–56), 281–309.

The author here gives a direct definition of the functor $\mathrm{Tor}_n^\Lambda(C, G)$ by generators and relations. To do this, he first introduces the general notion of the n-fold "slide product," $\langle C, n\Omega, G\rangle$, for a right module C, and left module G over a ring Ω. $\langle C, n\Omega, G\rangle$ is defined to be the abelian group generated by elements $\langle x_0, x_1, \cdots, x_n, x_{n+1}\rangle$ $x_0 \in C$, $x_i \in \Omega$, $x_{n+1} \in G$ such that $x_0x_1=0$, $x_1x_2=0$, $\cdots$, $x_{n-1}x_n=0$, $x_nx_{n+1}=0$ with the following relations:

$$\langle x_0, \cdots, x_iy, x_{i+1}, \cdots, x_{n+1}\rangle = \langle x_0, \cdots, x_i, yx_{i+1}, \cdots, x_{n+1}\rangle$$

$$\langle x_0, \cdots, x_i + x_i', \cdots, x_{n+1}\rangle = \langle x_0, \cdots, x_i, \cdots, x_{n+1}\rangle + \langle x_0, \cdots, x_i', \cdots, x_{n+1}\rangle$$

whenever both sides are defined.

For exact sequences, $0 \to A \to B \to C \to 0$, of right Ω-modules, connecting homomorphisms

$$\theta_n : \langle C, (n+1)\Omega, G\rangle \to \langle A, n\Omega, G\rangle$$

are defined, and it is shown that if every finitely generated left ideal of Ω is principal, and every finitely generated left Ω-module is a direct sum of cyclic modules, then $\{\langle C, n\Omega, G\rangle; \theta_n\}$ is an exact connected sequence of covariant functors [in the sense of Cartan and Eilenberg, Homological algebra, Princeton, 1956; MR **17**, 1040].

The author then shows that if Λ is any ring, and Ω is the ring of row- and column-finite infinite matrices over Λ, then Ω satisfies the above-mentioned conditions. Letting A^ω be the direct sum of a denumerable number of copies of A, where A is a right (left) Λ-module, the author proves that

$$\mathrm{Tor}_n^\Lambda(A, G) \approx \langle A^\omega, n\Omega, G^\omega\rangle \approx \langle A^\omega, n\Omega', G^\omega\rangle$$

where Ω' is the ring of all row-finite infinite matrices over Λ, and G^ω is the direct product of a denumerable number of copies of G. The above isomorphisms are natural.

D. A. Buchsbaum (Providence, R.I.).

Eilenberg, Samuel. Homological dimension and syzygies. Ann. of Math. (2) **64** (1956), 328–336.

Let $\Lambda = \Lambda^0 + \Lambda^1 + \Lambda^2 + \cdots$ be a graded ring. The radical N of Λ is defined as the intersection of all maximal homogeneous ideals in Λ. It is easily seen that $N = N^0 + \Lambda^1 + \Lambda^2 + \cdots$, where N^0 is the Jacobson radical of Λ^0. Throughout the paper it is assumed that $K = \Lambda^0/N^0$ is semi-simple and that each idempotent in K is the image of an idempotent in Λ^0. Let ${}_\Lambda\mathscr{M}$ be the category of all graded (left) modules and homomorphisms of degree zero of such modules. A subcategory $\mathscr{A}$ of ${}_\Lambda\mathscr{M}$ is said to be perfect if it satisfies the following axioms: 1) If $A, B \in \mathscr{A}$ and $\alpha: A \to B$, then $\alpha \in \mathscr{A}$; 2) if $\alpha: A \to B$ is an epimorphism and $A \in \mathscr{A}$, then $B \in \mathscr{A}$; 3) $\Lambda \in \mathscr{A}$; 4) if P is a projective module which is the direct sum of projective modules each of which is generated by a single homogeneous element and $P/NP \in \mathscr{A}$, then $P \in \mathscr{A}$; 5) if $A \in \mathscr{A}$ and $NA = A$, then $A = 0$.

Suppose $\mathscr{A}$ is a perfect subcategory of ${}_\Lambda\mathscr{M}$. An epimorphism $\varphi: P \to A$ in $\mathscr{A}$ is called minimal if P is projective and Ker $\varphi \subset NP$. If we assume that in addition to the above five axioms, $\mathscr{A}$ satisfies the sixth axiom that $A \in \mathscr{A}$ and $B \subset A$ implies that $B \in \mathscr{A}$, then given any $A \in \mathscr{A}$ one can construct in $\mathscr{A}$ a projective resolution

$$\cdots \to X_n \xrightarrow{d_n} X_{n-1} \to \cdots \xrightarrow{d_1} X_0 \xrightarrow{\epsilon} A \to 0$$

of A such that Im $d_{i+1} \subset NX_i$ for all $i = 0, 1, \cdots$. Projective resolutions of A which satisfy this condition are called minimal resolutions. It is shown that any two minimal resolutions of A are isomorphic and that any projective resolution Y of A is the direct sum of complexes X and W where X is a minimal resolution of A and W is a projective resolution of the module 0. Using these notions the author proves that if each category ${}_\Lambda\mathscr{M}$ and $\mathscr{M}_\Lambda$ contains a perfect subcategory satisfying axiom 6 and n is an integer such that $\mathrm{Tor}_n(K, K) \neq 0$ and $\mathrm{Tor}_{n+1}(K, K) = 0$, then $\mathrm{l.dim}\,{}_\Lambda K = \mathrm{gl.dim}\,{}_\Lambda\mathscr{M} = n = \mathrm{gl.dim}\,\mathscr{M}_\Lambda = \mathrm{r.dim}_\Lambda K$, where $\mathrm{gl.dim}\,{}_\Lambda\mathscr{M} = \sup \mathrm{l.dim}\,{}_\Lambda A$ for all $A \in {}_\Lambda\mathscr{M}$.

The paper concludes with a discussion of graded rings Λ for which the category ${}_\Lambda\mathscr{M}$ contains perfect subcategories satisfying axiom 6. The examples given are 1) Λ^0 is a left Noetherian ring and each Λ^n is finitely generated as a left Λ^0-module; 2) N^0 is nilpotent. *M. Auslander.*

Henriksen, Melvin. On the equivalence of the ring, lattice, and semigroup of continuous functions. Proc. Amer. Math. Soc. 7 (1956), 959–960.

Shirota [Osaka Math. J. 4 (1952), 121–132; MR 14, 669] showed that a Q-space is determined by its lattice or by its multiplicative semigroup of continuous real functions. The author points out that this settles once for all the equivalence of the following three statements concerning two topological spaces: their rings (lattices, semigroups) of continuous functions are isomorphic.

I. Kaplansky (Princeton, N.J.).

See also: Higgins, p. 559; McCarthy, p. 562; Carlitz and Uchiyama, p. 563; Jennings and Ree, p. 583; Strickler, p. 591; Gheorghiu, p. 594.

Groups, Generalized Groups

Heerema, Nickolas. Sums of normal endomorphisms. Trans. Amer. Math. Soc. 84 (1957), 137–143.

Let G be a group with center Z_G and commutator subgroup C_G. Let N, A, and E represent the sets of normal nilpotent endomorphisms, normal automorphisms and normal endomorphisms of G respectively. Finally, let K_G represent the system of mappings of G generated by the elements of E with respect to the usual sum and product of endomorphisms. It is well-known that, if G is commutative, then E is a ring with respect to sum and product. Fitting [Math. Ann. 107 (1932), 514–542] investigated the structure of E in the case of a non-abelian G. The author here studies K_G. The following are representative theorems: (1) K_G is a ring. (2) If $\alpha \in A$ and $\beta \in E$, then $\alpha-\beta \in E$. If also $\beta \in A$, then $(\alpha-\beta)(G) \subset Z_G$. If $\beta \in N$ and $\alpha \in E$, then $\alpha-\beta \in E$. If also $\alpha \in N$, then $(\alpha-\beta)(G) \subset Z_G$. (3) If G is non-commutative, satisfies both chain conditions, and has no non-trivial direct abelian factor, then N is a group with respect to addition; $\alpha \in A$ and $\beta \in N$ implies $\alpha-\beta \in A$ and $\alpha+\beta \in A$; $\alpha \in A$ and $\beta \in A$ implies $\alpha-\beta \in N$; if G is indecomposable then K_G is generated by the identity map and N. (4) A normal mapping γ on a non-abelian group G is a sum of n normal endomorphisms each of which leaves C_G elementwise fixed if and only if γ has the following properties: (I) $\gamma(x)^{-1}y\gamma(x)=x^{-n}yx^n$, (II) $\gamma(xy)=\gamma(x)\gamma(y)[y^{-n}x^{-n}(xy)^n]$. (5) If G satisfies both chain conditions and is indecomposable, γ belongs to K_G if and only if γ or $-\gamma$ satisfies relations I and II for some $n \geqq 0$.

R. L. San Soucie.

Higgins, P. J. Groups with multiple operators. Proc. London Math. Soc. (3) 6 (1956), 366–416.

The groups with multiple operators form a class of abstract algebras which includes e.g. groups, rings, linear algebras, Lie and Jordan triple systems. The paper extends fundamental theorems of group-theory — in particular those about ascending and descending series and about direct decomposition — to this class of abstract algebras. For suitable definitions and after proper adaption of the proofs, the arguments used in group-theory still hold in this wider field. Moreover, it appears that the associative law of group-composition plays a minor rôle only and therefore most of the theory remains valid for loops with multiple operators.

Let G be an additive (not necessarily commutative) group, $\mathbf{a}=(a_1, \cdots, a_n)$ an ordered set of elements of G; an operator ω of weight n maps $\mathbf{a} \to \mathbf{a}\omega \in G$ with $0 \cdots 0\omega = \mathbf{0}\omega = 0$. Every "group G with multiple operators" (Ω-group) is closed for a set Ω of operators ω; every subset of G which is closed for $+$, $-$, and Ω, is called an Ω-subgroup of G. If in particular all the operators are of weight 1 (unary) and are endomorphisms, G is an ordinary group with operators; if G is a ring, the multiplication is defined by a binary operator etc. Starting from a system X of indeterminates x_i and applying repeatedly $+$, $-$, Ω on the finitary subsets of X, one obtains the free Ω-words $f(\mathbf{x})$ over X; correspondingly $f(\mathbf{x}, \mathbf{y})$ for 2 systems X, Y of indeterminates. When the well-known identities of the theory of free groups are admitted, the words $f(\mathbf{x})$ form the "free Ω-group over X". By substituting $\mathbf{a}$ for $\mathbf{x}$, one gets $f(\mathbf{a}) \in G$. Suppose that X and Y are distinct sets of indeterminates, $a_i \in A \subseteq G$, $b_j \in B \subseteq G$, then the subset of the elements $f(\mathbf{a}, \mathbf{b})$, for all the $f(\mathbf{x}, \mathbf{y})$ satisfying $f(\mathbf{0}, \mathbf{y})=0$, is called the transform $A^B \supseteq A$ of A by B. If in particular H is an Ω-subgroup of G and $H^G \subseteq H$, then $H^G=H$ and is called an "ideal". The ideals correspond to the normal subgroups of group-theory. The cosets of H form an Ω-group G/H. If $f(\mathbf{x}, \mathbf{y})$ satisfies $f(\mathbf{0}, \mathbf{y})=f(\mathbf{x}, \mathbf{0})=0$, the elements $f(\mathbf{a}, \mathbf{b})$ form an Ω-subgroup $[A, B] \subseteq A^B \cap B^A$ called the commutator Ω-subgroup of A and B. If G is a group, $[G, G]$ is the ordinary commutator-group; if G is a ring, $[A, B]$ is the Ω-subgroup generated by the elements a_ib_k. The abelian groups, defined by $[G, G]=0$, coincide in the case of groups with the commutative groups and in the case of rings with the zero-rings. These notions are applied to generalise the theorems concerning the lattice of the normal subgroups of a group and their successive normal subgroups to the lattice of the ideals of G and the subideals of any order. Besides projectivity, "Z-projectivity" is introduced in this lattice. Two serial factors K/H and K'/H' are called Z-projective if $K \cap K'/H \cap H'$ is a reduction of each of them. If G possess a composition-series and 2 composition-factors are projective, either they are Z-projective, or they are abelian. A mapping α of G into itself such that $f(0, y)=0$ implies $f(g, h)\alpha = f(g\alpha, h)$, for all $g, h \in G$, is necessarily an endomorphism and is said to be a normal endomorphism in generalisation of this notion in group-theory. The complement β of a normal endomorphism α is normal; $\alpha\beta=\beta\alpha$ is a central endomorphism and $[G\alpha, G\beta]=0$. With the help of these notions, the theory of group-decomposition is generalised to the Ω-groups.

F. W. Levi (Berlin).

Abhyankar, Shreeram. On the finite factor groups of abelian groups of finite rational rank. Amer. J. Math. 79 (1957), 190–192.

Theorem: If G is a torsion-free abelian group of rank n, then any finite homomorphic image of G is a direct sum of n or fewer cyclic groups. "A shorter proof is possible, based on the fact that any finitely generated subgroup of G can be generated by n elements, and that this property is preserved under homomorphism." (Letter from the author to the reviewer.) In a previous paper [same J. 78 (1956), 761–790; MR 18, 600] the author showed that Theorem Z (a conjectural form of the resolution problem due to Zariski) must be weakened at least so as to replace twofold crossings by s-fold crossings with $s \leqq$ dimension of the variety. The consequence of the above theorem is that as far as valuation-ramification theory is concerned this weakened form is satisfactory.

I. Kaplansky.

Stojaković, Mirko. Sur une relation d'ordre dans le groupe symétrique. Acad. Serbe Sci. Publ. Inst. Math. 10 (1956), 71–78.

An order relation amongst the elements of S_n is defined

which facilitates the construction of the multiplication table. *G. de B. Robinson* (Toronto, Ont.).

Osima, Masaru. Note on a paper by J. S. Frame and G. de B. Robinson. Math. J. Okayama Univ. 6 (1956), 77–79.

This paper gives an analysis of p-regular and p-singular Young diagrams on the basis of integers $\rho_j=\alpha_j'-\alpha_{j+1}'$, $\rho_k=\alpha_k'$, where α_i' is the number of nodes in the ith column of $[\alpha]$. Clearly, $[\alpha]$ has no p-rows of equal length (i.e., is p-regular) if and only if $\rho_j<p$; $[\alpha]$ is p-singular in the contrary case when at least one $\rho_j\geq p$ (note the misprint in line 2 p. 78). Thus in the p-singular case we may set

$$\rho_j=\rho_j^{(1)}+\rho_j^{(2)}p \quad (0\leq\rho_j^{(1)}<p),$$

so that the $\rho_j^{(1)}$'s define a regular diagram $[\alpha^{(1)}]$, and $[\alpha]$ and $[\alpha^{(1)}]$ have the same p-core. Moreover, $[\alpha^{(1)}]$ and any diagram $[\alpha^{(2)}]$ define a unique $[\alpha]$ so that the basis of an induction is established and it follows easily that the number of p-regular diagrams in a block is equal to the number of modularly irreducible representations in the block. *G. de B. Robinson* (Toronto, Ont.).

★ **Fischer, Emil. Einführung in die geometrische Kristallographie.** Akademie-Verlag, Berlin, 1956. viii+164 pp. (12 plates). DM 23.00.

After showing how crystals are measured, the author describes the six crystal systems (which later become seven, when the trigonal system is subdivided into trigonal-rhombohedral and hexagonal). He illustrates each system by drawings of typical minerals and of the appropriate frame of coordinate axes. He then treats lattices, rational indices, zones, duality, and stereographic projection. His analysis of the symmetry of the cube leads naturally to a general discussion of rotations, reflections, rotatory-reflections and rotatory-inversions. After proving that the only possible rotational symmetry-operations of a lattice are of periods 2, 3, 4, 6, he enumerates the 32 crystal classes [cf. W. Barlow, Phil. Mag. (6), 1 (1901), 1–36; H. S. M. Coxeter, Regular polytopes, Methuen, London, 1948, pp. 53–55; MR 10, 261; G. Polya and B. Meyer, C. R. Acad. Sci. Paris 228 (1949), 28–30; MR 10, 281; H. Weyl, Symmetry, Princeton, 1952, pp. 149–156; MR 14, 16]. He then shows how the 32 classes are distributed in the six or seven systems, and how the system and class of a given crystal can be recognized. The book is enlivened by excellent figures and tables, many exercises (with answers at the end), and a dozen photographs of natural crystals. *H. S. M. Coxeter.*

See also: Artin, p. 553; Petropavlovskaya, p. 555; Jennings and Ree, p. 583; Phillips, Rogers and Wilson, p. 605; Wilson, p. 605.

THEORY OF NUMBERS

General Theory of Numbers

Carlitz, L. A note on the Staudt-Clausen theorem. Amer. Math. Monthly 64 (1957), 19–21.

Put $\log\{(1-e^{-x})/x\}=\sum_{n=1}^{\infty} R_n x^n/n!$ so that $R_n=B_n/n$, where B_n is the nth Bernoulli number. Theorem. Let p^r denote the highest power of the prime p dividing $2m$. Then if $p-1\nmid 2m$, R_{2m} is integral (mod p). If $p-1|2m$, then

$$p^{r+1}R_{2m}\equiv p^r(p-1)/2m \pmod{p^{r+1}} \quad (p\geq 3),$$

and

$$2^{r+1}R_{2m}\equiv 2^r/2m \pmod{2^{r+1}} \quad (p=2,\ m\geq 2).$$

The proof is based upon the Staudt-Clausen theorem and related results. *A. L. Whiteman.*

Hornfeck, Bernhard. Bemerkung zu meiner Note über vollkommene Zahlen. Arch. Math. 7 (1956), 273.

The result of an earlier paper [Arch. Math. 6 (1955), 442–443; MR 17, 460] is sharpened to show that $\limsup V(x)/x^{\frac{1}{4}}\leq 1/(2\sqrt{5})$, where $V(x)$ is the number of perfect numbers not exceeding x. *R. A. Rankin.*

Cohen, Eckford. Some totient functions. Duke Math. J. 23 (1956), 515–522.

This is a sequel to two earlier papers of the author [same J. 16 (1949), 85–90; 22 (1955), 543–550; MR 10, 354; 17, 238]. It is mainly concerned with elementary identities involving the function $\phi_k(n)$, this being defined as the number of integers a with $0\leq a<n^k$ such that (a, n^k) has no kth power divisor (>1). *H. Davenport* (London).

Schäffer, Juan Jorge. A result in elementary number theory. Nieuw Arch. Wisk. (3) 4 (1956), 118–123.

The following result, which is related to earlier work on Diophantine equations and Bernoulli polynomials [Acta Math. 95 (1956), 155–189; MR 17, 1187], is proved: — Let n be a positive integer. Let

$$n=\sum_{i=0}^{t(p)} n_{ip}p^i \quad (0\leq n_{ip}<p),$$

where p is an odd prime, be the expression of n in the scale of p. Suppose that

$$\sum_i n_{ip}<p$$

for every odd p. Then n is one of the numbers 1, 2, 3, 4, 6, 10, 12, 28, 30, 36. Indeed the result holds if p is restricted to the values 3, 5, 7, 11, 13. *J. W. S. Cassels.*

Kazarinoff, Donat K. On Wallis' formula. Edinburgh Math. Notes no. 40 (1956), 19–21.

Wallis' formula is usually written

$$\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}=\frac{1}{\sqrt{\{\pi(n+\theta)\}}} \quad (0<\theta<\tfrac{1}{2};\ n=1, 2, \cdots).$$

In this note it is shown that $\frac{1}{4}<\theta<\frac{1}{2}$.

Hall, Marshall, Jr. A survey of difference sets. Proc. Amer. Math. Soc. 7 (1956), 975–986.

A set D of k distinct residues $d_1, d_2, \cdots, d_k$ modulo v is a difference set provided every residue $b\not\equiv 0 \pmod v$ can be written in exactly λ ways in the form $d_i-d_j\equiv b \pmod v$, with d_i, d_j in D. This paper makes a survey of all difference sets with parameters v, k, λ, with k in the range $3\leq k\leq 50$. The parameters must satisfy $\lambda=k(k-1)/(v-1)$ and k may be restricted without loss of generality by $k<v/2$. There are then 268 choices of parameters for difference sets. Of these, 101 are excluded by the criterion of Chowla and Ryser for symmetrical designs. Of the remaining 167 choices, difference sets were found in 46 cases. Of these 46 cases, in three there are two non-isomorphic soluti nons and ione case, $v=121$, $k=40$, $\lambda=13$,

there are four non-isomorphic solutions. In 109 cases, difference sets were shown not to exist, and in only 12 cases the existence problem remains undecided.

The survey utilizes hand computation, the electronic computer SWAC, and the extensive theory that has been developed for difference sets in recent years. The paper also contains two theorems that are new. The first is a generalization of the multiplier theorem. Let n_1 divide $n=k-\lambda$ and let $(n_1, v)=1$ and $n_1>\lambda$. Let t be an integer such that for each prime divisor p of n_1, there is a j such that $p^j\equiv t \pmod v$. Then t is a multiplier of the difference set $d_1, d_2, \cdots, d_k \pmod v$. The second of the theorems was suggested by computations involving difference sets of the Hadamard variety, and characterizes a set of residues forming a difference set modulo a prime $p=6f+1$ which includes the sextic residues as multipliers. *H. J. Ryser.*

Straus, E. G.; and Swift, J. D. The representation of integers by certain rational forms. Amer. J. Math. 78 (1956), 62–70.

The authors consider the Diophantine equations

$$(1)\qquad f(x)=z, \text{ where } f(x)=N(x)/D(x)$$

and $N(x)$ and $D(x)$ are polynomials in $x=(x_1, x_2, \cdots, x_n)$ of degree no higher than 2 in each x_i and the degree of $D(x)$ is at least as great as that of $N(x)$. If $D_1(x)$ is the homogeneous polynomial consisting of the terms of highest degree of $D(x)$ they call the locus $D_1(x)=0$, the critical cone $\mathscr{C}$. A conical neighborhood of $\mathscr{C}$ is an open set of rays originating at the origin and containing $\mathscr{C}$. A conjugate lattice point of $(x_1, x_2, \cdots, x_n, z)$ is obtained by replacing one x_i for which (1) is quadratic in x_i by x_i' the conjugate of x_i when all the other x_i in (1) are considered fixed.

The authors prove: If every lattice point x satisfying (1) has a conjugate lattice point in the exterior of some conical neighborhood of the critical cone, then (1) has solutions for at most a finite number of z. The cases $n=2$ and $n=3$ are considered in more detail and possible extensions indicated. *B. W. Jones* (Boulder, Colo.).

Thébault, Victor. Sur des suites de Pell. Mathesis 65 (1956), 390–395.

The paper starts from the following known property of the Pell equation: If we form the two sequences $x_{n+1}=2x_n+1x_{n-1}$, $x_0=1$, $x_1=1$ and $y_{n+1}=2y_n+1y_{n-1}$, $y_0=0$, $y_1=1$, then (x_{2k}, y_{2k}) yield all solutions of

$$x^2-2y^2=+1,$$

and (x_{2k}, y_{2k}) all solutions of $x^2-2y^2=-1$. Several relations between the x's and y's are established by elementary considerations, for example

$$y_{n+1}^2-y_n^2-2y_n\cdot y_{n+1}=(-1)^n;$$

$$x_{n+1}+x_n=2y_{n+1};\ y_n\cdot y_{n+1}-y_{n-1}\cdot y_{n+2}=2\cdot(-1)^{n-1}$$

(misprint in paper);

$$x_{n-1}\cdot x_n\cdot x_{n+1}\cdot x_{n+2}+4=(x_n\cdot x_{n+1}+2\cdot(-1)^{n+1})$$

(misprint in paper); etc. Similar types of relations are established for the sequences $y_{n+1}=p\cdot y_n+1\cdot y_{n-1}$, p a positive integer. For some reason the importance of this sequence for Pell's equation (for suitably chosen p) is not mentioned. *A. J. Kempner* (Boulder, Colo.).

Hampel, R. On the solution in natural numbers of the equation $x^m-y^n=1$. Ann. Polon. Math. 3 (1956), 1–4.

The author proves that the equations

$$n^{\alpha+s}-(n+1)^\alpha=\pm 1$$

with $n\geqq 2$, $\alpha\geqq 2$, $s\geqq 1$ are not solvable except in the trivial case $n=2$, $\alpha=2$, $s=1$. The case with $+1$ on the right hand side is readily reduced to the solubility of $2^{\alpha+s}-3^\alpha=1$, and it is then not difficult to see that the expansions of 3^α and $2^{\alpha+s}-1$ in the scale of 2 always fail to agree in the last three digits. The case with -1 on the right is more difficult, but the author disposes of it by showing that the representation of $(n+1)^\alpha$ in the scale of n cannot, except in the trivial case, have as many zero digits as $n^{\alpha+s}+1$. The paper concludes with a slight generalisation. *H. Halberstam* (Exeter).

Schinzel, A. Sur l'équation $x^z-y^t=1$, où $|x-y|=1$. Ann. Polon. Math. 3 (1956), 5–6.

The author gives a shorter proof of the theorem reviewed above by a method based on primitive roots and a result of B. A. Hausmann [Amer. Math. Monthly 48 (1941), 482] which states that 2^m+1 for $m>3$ cannot be the power of an integer with exponent greater than 1. *H. Halberstam* (Exeter).

Rotkiewicz, A. Sur l'équation $x^z-y^t=a^t$, où $|x-y|=a$. Ann. Polon. Math. 3 (1956), 7–8.

A theorem of G. D. Birkhoff and Vandiver [Ann. of Math. (2) 5 (1904), 173–180] states that if a, b, n are natural numbers, $a>b$, $(a, b)=1$, $n>2$, then a^n-b^n is divisible by at least one prime p such that p does not divide any of the integers a^r-b^r $(r=1, 2, \cdots, n-1)$; the case $a=2$, $b=1$, $n=6$ provides the sole exception. From this the author concludes, by a very short and simple argument, that the equation $x^z-y^t=a^t$, where a is an integer, has no integer solutions x, y, z, t greater than 1, other than $x=3$, $y=2$, $z=2$, $t=3$, such that $|x-y|=a$ and $(x, y)=1$. If $a=1$, we arrive again at the theorem discussed in the preceding two reviews. *H. Halberstam* (Exeter).

Ballantine, J. P. Integral approximate solutions of systems of linear equations. Amer. Math. Monthly 63 (1956), 554–569.

The author considers the solution of several types of the equation $\sum_{j=1}^n a_{ij}x_j=b_i$ $(i=1, 2, \cdots, n)$. The equations to be solved are either exact, approximate, integral, least square or combinations of these. The author describes an approximate solution where the unknowns have integral value, i.e., type *(ia)* and similarly a type *(lia)*. The latter signifies a type *(ia)* solution that is compared with other *(ia)* solutions and has the sum of the squares of its deviation a minimum. In order to obtain a type *(lia)* solution, recourse is made to the quadratic form approach proposed originally by Minkowski. Applying the aforementioned method, the author deftly suggests and demonstrates the utility of this approach. The procedure is unique in that a computer can initiate and continue with the solution of the above mentioned types by following the procedure as outlined in the sample examples. It is not necessary for him to understand the necessary mathematical proofs. *H. Saunders* (Philadelphia, Pa.).

Mouette, L. Sur la théorie des formes quadratiques binaires. Mathesis 65 (1956), 364–371.

The author gives the usual definitions of equivalence of binary quadratic forms and, using the theory of composition, points out certain obvious consequences of representations of numbers by forms of given determinant d. If $h(d)$ denotes the class number and

$$\mu'(d) \text{ or } \mu''(d) = h(d)/\sqrt{|d|}$$

according as d is negative or positive. He gives a table of values of these functions for square-free values of d from 2 to 57, 1001 to 1055. He tabulates $\sum \mu'$ and $\sum \mu''$ over the first 100 values of d, the second 100, up to the fourteenth 100 and observes that the respective mean values seem to approach .754 and .997. (These results seem to be allied to Merten's asymptotic value of $\sum h(d)$.) He lists the determinants of positive forms having one and two classes in a genus; and the number having three and four classes presumably up to the limits of his table.
B. W. Jones (Boulder, Colo.).

McCarthy, Paul J. Representation by quadratic forms in valuation rings. Portugal. Math. **15** (1956), 1–7.

Let K be a field which is complete with respect to a non-Archimedean valuation, and let R be the valuation ring of K with respect to this valuation. On the assumption that the residue class field of K is not of characteristic 2, the author uses results of the reviewer and W. H. Durfee to derive theorems on the representation of a form g in m variables by a form f in n variables, with $n \geqq m$. He shows that the classical results hold, that is: if f is a form of unit determinant in R, then f represents a number N in R if and only if it represents N in K. Also if f and g are two forms in R whose determinants are units in R, f represents g in R if and only if it represents g in K, for $n \geqq m$.
B. W. Jones (Boulder, Colo.).

O'Meara, O. T. Integral equivalence of quadratic forms in ramified local fields. Amer. J. Math. **79** (1957), 157–186.

The author considers the problem of finding a set of invariants of a quadratic form which completely characterizes an equivalence class. This has been accomplished by W. H. Durfee, B. W. Jones, and Gordon Pall over local fields in which 2 is a unit and over p-adic numbers. The author solved the problem for any local field in which 2 is a prime and for unitary forms in the ramified extensions of 2-adic numbers [same Amer. **77** (1955), 87–116; MR **16**, 680]. This paper considers the remaining case-fields where 2 is not a unit. Some simplifications of earlier proofs result. The invariants are too complex to give here.
B. W. Jones (Boulder, Colo.).

Piehler, Joachim. Zur Theorie der binären kubischen Formen. Math. Ann. **132** (1956), 177–179.

The author considers the primitive cubic form

$$f = ax^3 + bx^2y + cxy^2 + dy^3$$

with integer coefficients whose g.c.d. is 1. He denotes by T the g.c.d. of the coefficients of its Hessian covariant form. For a prime p of the form $3n+1$ and K the multiplicative group of cubic residues mod p, the factor group R_p/K is of order 3. When all numbers prime to p represented by f lie in the same residue class of R_p/K, f is said to have a character (mod p). The author proves that f has a character (mod p) for p of the form $3n+1$ if and only if p is a divisor of T.
B. W. Jones.

McCarthy, Paul J. The existence of indefinite ternary genera of more than one class. Duke Math. J. **24** (1957), 19–24.

Let f be an indefinite ternary quadratic form whose matrix, A, has integral elements. In the classical notation, let Ω be the g.c.d. of the two-rowed minor determinants of A and Δ is defined by $|A| = \Omega^2\Delta$. The author proves the following generalization of results of the reviewer and Hadlock [Proc. Amer. Math. Soc. **4** (1953), 539–543; MR **15**, 106]. Theorem 1. Let $\Omega = 2^m\Omega_1{}^2\pi$ and $\Delta = -2^\delta\Delta_1{}^2\pi$, where Ω_1 and Δ_1 are odd and π is an odd prime. Let ω and δ be both even and $\omega+\delta \geqq 2$ if $\pi \equiv 5 \pmod 8$, and $\omega+\delta \geqq 4$ if $\pi \equiv 3 \pmod 4$. Then there are at least two properly primitive forms with properly primitive reciprocal forms, which have the given invariants and which are in the same genus but not the same class.

Using related forms the author proves the same result for the same conditions as given above except that $\Delta = -2^\delta\Delta_1{}^2$ with Δ_1 divisible by π.
B. W. Jones.

van der Waerden, B. L. Die Reduktionstheorie der positiven quadratischen Formen. Acta Math. **96** (1956), 265–309.

This paper is a long-needed exposition, simplification, and unification of the reduction theory of positive quadratic forms, $f(x_1, x_2, \cdots, x_n) = \sum f_{ij}x_ix_j$. Chief attention is given to two definitions of reduced form. First, Hermite called a form reduced if $f_{ii} = N_i$ where the N_i are the "successive minima" of the form; that is, N_1 is the least positive value of f_{11} for all forms equivalent to f, N_2 is the least positive value of f_{22} for all forms equivalent to f and having $f_{11} = N_1$, etc. Second, Minkowski called a form f reduced if the following condition holds:

(D) $$f_{kk} \leqq f(s_1, s_2, \cdots, s_n)$$

for all integers $s_1, \cdots, s_n$ with $(s_k, \cdots, s_n) = 1$. The fundamental inequalities are the following for the Minkowski reduced form

(E) $$\lambda_n f_{11} f_{22} \cdots f_{nn} \leqq D_n,$$

and the following for the successive minima:

(F) $$N_1 N_2 \cdots N_n \leqq \mu_n D_n.$$

After laying the basis for the reduction theory and proving inequality (F), the author sharpens Mahler's inequality, $f_{kk} \leqq \delta_k N_k$, to deduce inequality (E) with Remak's coefficient. He gives proofs of the two "Finiteness Theorems": For a given determinant, there is only a finite number of forms for which one or more equalities in (D) hold; there is only a finite number of integral transformations taking reduced forms into reduced forms. Special attention is given to binary, ternary, and quaternary forms.

In the second part of this paper, the author considers the geometrical phase of the reduction theory in which the set of reduced forms form a cell in the $\frac{1}{2}n(n+1)$ dimensional space of positive quadratic forms. The space is covered by these cells with overlapping on the boundaries.

The third part of the paper deals with "Extreme Forms" such that for each small variation of the variables which leaves fixed the first minimum, the discriminant increases. He proves the theorem of Korkine and Zolotareff: If f is an extreme form, the set of forms having its first minimum forms a linear subspace of the space of all forms of given discriminant. Again special attention is given to ternary and quaternary forms and a brief survey given of the results of Korkine, Zolotareff, Hofreiter, and Coxeter.
B. W. Jones (Boulder, Colo.).

Kneser, Martin. Klassenzahlen indefiniter quadratischer Formen in drei oder mehr Veränderlichen. Arch. Math. **7** (1956), 323–332.

M. Eichler defined spinor genera and showed that a spinor genus of indefinite quadratic forms in more than two variables contains only one class [Quadratische For-

men und orthogonale Gruppen, Springer, Berlin, 1952; Math. Z. **55** (1952), 216–252; MR **14**, 540]. Using a slightly different definition of genus from the classical one, the author shows that the number of spinor genera in a genus can be written as a certain group index. Roughly speaking, if the reduced determinant D of the form has only a few prime divisors to high powers, the number of genera in a spinor genus is small — in particular it is 1 if D has no cubic factors. More generally (and more precisely) if R is an indefinite space of dimension greater than 2 over the field of rational numbers, if the reduced determinant of the lattice is equal to $\prod_p p^{s_p}$ with

$$s_2<\tfrac{1}{2}n(n-1)+[\tfrac{1}{2}(n+1],\ s_p<\tfrac{1}{2}n(n-1)$$

for odd p, the genus of K has but one class.

B. W. Jones (Boulder, Colo.).

See also: Ramanathan, p. 555; Erdös, p. 563.

Analytic Theory of Numbers

Tanaka, Minoru. On the number of prime factors of integers. Jap. J. Math. **25** (1955), 1–20 (1956).

The author proves (among others) the following theorem: Let $f_i(x)$ $(1\leqq i\leqq k)$ be a set of non-constant polynomials which are pairwise relatively prime. Assume that $f_i(x)$ is the product of r_i irreducible polynomials (multiple factors are counted only once). Let E be any Jordan measurable set in k dimensional space and $V(n)=\sum_{p|n} 1$. $A(x;E)$ denotes the number of integers $n<x$ for which the point $(\mu_1(n), \mu_2(n), \cdots, \mu_k(n))$ belongs to E, where

$$\mu_i(n)=\frac{V(f_i(n))-r_i\log\log n}{\sqrt{(r_i\log\log n)}}.$$

Then

$$\lim_{x\to\infty}\frac{A(x;E)}{x}=\frac{1}{(2\pi)^{k/2}}\int_E\exp\Big(-\tfrac{1}{2}\sum_{i=1}^{k}\mu_i^2\Big)\,d\mu_1\cdots d\mu_k.$$

This generalizes previous results of Hardy-Ramanujan, Turán, Erdös-Kac, Delange and others [cf. also some recent results of Leveque, Trans. Amer. Math. Soc. **66** (1949), 440–463; MR **11**, 83; and Halberstam, J. London Math. Soc. **30** (1955), 43–53; **31** (1956), 1–14, 14–27; MR **16**, 569; **14**, 461].

The author discusses several interesting special cases of his theorem.

P. Erdös.

Erdös, P. On perfect and multiply perfect numbers. Ann. Mat. Pura Appl. (4) **42** (1956), 253–258.

Let $P(x)$ denote the number of perfect and multiply perfect numbers not exceeding x. Then it is shown that

$$P(x)<x^{\varepsilon+3/4}$$

for every $\varepsilon>0$ and for all sufficiently large x. The proof uses the fact that if $\sigma(n)$ is the sum of all the divisors of n then

$$\limsup_{n=\infty}\sigma(n)/[n\log\log n]=e^C$$

for Euler's constant C so that $\sigma(n)<2n\log\log n$ for n sufficiently large. The contributions of square-free and non square-free factors of n are considered separately. A second theorem with a similar proof states that there are less then $x^{\frac{1}{2}-c}$ perfect numbers not exceeding x for some constant $c>0$ and for all $x>x_0$. For the more general case in which $\sigma(n)$ and n have a common factor the following result is stated without proof: The density of integers n for which $\sigma(n)$ and n have a greatest common divisor less than $(\log\log n)^\alpha$ is an increasing function of α ranging between 0 and 1.

D. H. Lehmer.

Carlitz, L.; and Uchiyama, S. Bounds for exponential sums. Duke Math. J. **24** (1957), 37–41.

Let k be the Galois field with $q=p^n$ elements (p prime). The equation $y^p-y-F=0$, where $F\in k[x]$ and is chosen such that the left hand side is absolutely irreducible, determines a cyclic extension $\Omega=K(y)$ of $K=k(x)$ of degree p. The zeta function of Ω can be written in the form $(1-q^{1-s})^{-1}\prod_{u=1}^{p-1}L(s,\lambda^u)$, where the $L(s,\lambda^u)$ are polynomials $\sum_{j=0}^{r-1}f_jX^j$ in $X=q^{-s}$ where r is the degree of the polynomial F. By the Riemann hypothesis the roots of these polynomials in X have absolute value $q^{-\frac{1}{2}}$ and it follows that $|f_1|\leqq(r-1)q^{\frac{1}{2}}$. This inequality can be written in the form (t denotes the trace in k)

$$|\sum_{\alpha\in k}\exp 2\pi it(F(\alpha)/p|\leqq(r-1)q^{\frac{1}{2}}.$$

The same method gives also

$$|\sum_{\alpha\in k}\exp 2\pi it(\alpha+c\alpha^{-1})/p|\leqq 2q^{\frac{1}{2}}\ (\alpha\neq0,\ c\neq0).$$

These estimates are thus obtained from the Riemann hypothesis for function fields in a more elementary way than in Weil's paper [Proc. Nat. Acad. Sci. U.S.A. **34** (1948), 204–207; MR **10**, 234]. The authors give also an application to some results by Uchiyama [Proc. Japan Acad. **31** (1955), 199–201; **32** (1956), 97–98; MR **17**, 130, 937].

H. D. Kloosterman (Leiden).

Postnikov, A. G. On the generalization of one of the Hilbert problems. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 512–515. (Russian)

Soit χ un caractère modulo m, et posons $L(x,s,\chi)=L_x=\sum\chi(n)x^nn^{-s}$. L'auteur démontre le théorème suivant: Entre differentes fonctions L_x (modulo m) aucune relation de la forme

$$\Phi\Big(x,s,\frac{\partial^{p+q}L(x,s,\chi)}{\partial x^p\partial s^q}\Big)=0,$$

où Φ est un polynome, n'est possible. Ce théorème généralise un theorème d'Ostrowski correspondant aux fonctions du type ξ, et qui est une réponse à un des problèmes posés par Hilbert au Congrès de Paris [voir Ostrowski, Math. Z. **8** (1920), 241–298]. L'auteur utilise plusieurs faits étables par Ostrowski dans le travail cité, ainsi que le lemme suivant: En posant $L=L(s,\chi)=\sum\chi(n)n^{-s}$, la rélation $\Phi(s, f_k^{(\nu)}(s+h_{k,\nu,\tau}))=0$, où f_k sont differentes fonctions L (modulo m) et où $h_{k,\nu,\tau}$ sont des constantes, est impossible.

S. Mandelbrojt (Paris).

Rodosskiĭ, K. A. On the exceptional zero. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 667–672. (Russian)

The author considers the real zeros of L-functions formed with real primitive residue characters modulo $D>100$. For fixed $\varepsilon\in(0;0.025]$ it is shown that there is at most one zero β of one exceptional L-function of this class which fails to satisfy

$$1-\beta\geqq\min\{\varepsilon; 0{\cdot}015\ \ln^{-5}D\cdot D^{-26\varepsilon}\}.$$

A consequence is that, without exception,

$$1-\beta\geqq C(\varepsilon)\cdot D^{-30\varepsilon},$$

effectively a result due to Siegel [see, e.g., S. Chowla, Ann. of Math. (2) 51 (1950), 120–122; 11, MR 420].
F. V. Atkinson (Canberra).

Marinina, S. F. Estimation of the number of irregular L-functions of a quadratic field. Ukrain. Mat. Ž. 8 (1956), 319–324. (Russian)

Let K be an imaginary quadratic field of class number h and discriminant $-d$. Suppose that

$$\varepsilon>0, \frac{5}{6}\leqq\Delta<1, -\infty<T<\infty, T_1=|T|+\frac{\sqrt{5}}{2}.$$

It is shown that there is a constant $c(\varepsilon)$, depending only on ε, with the following property: if Q is the number of L-functions over K having at least one zero in the region $\Delta\leqq\sigma\leqq1$, $|t-T|\leqq\frac{1}{2}$, then

$$Q<c(\varepsilon)(T_1h^2d^{1/2})^{6(1+\varepsilon-\Delta)/(2\Delta-1)}.$$

The proof depends on weighted character sums, which were used earlier by K. A. Rodosskiĭ [Ukrain. Mat. J. 3 (1951), 399–403; MR 15, 202] and I. P. Kubilyus [Mat. Sb. N.S. 31(73) (1952), 507–542; MR 14, 847].
W. J. LeVeque (Ann Arbor, Mich.).

Rodosskiĭ, K. A. On non-residues and zeros of L-functions. Izv. Akad. Nauk SSSR. Ser. Mat. 20 (1956), 303–306. (Russian)

Let $N_{\min}(D, k)$ be the least positive non-residue of exponent k for prime modulus D. Let the function $L(s, x)$, with non-principal character of exponent k for prime modulus $D>D_0$, not vanish in the rectangle

$$1-\Psi\ln^{-1}D\leqq\sigma\leqq1,\ |t|\leqq\min\{e^{\Psi}\ln^{-1}D, 1\},$$

where $\Psi\in[\varepsilon, \frac{1}{8}\ln D]$. Then

$$N_{\min}(D, k)<D^{A\Psi^{-1}\ln\Psi},$$

where D_0 and A are absolute positive constants.
N. Levinson (Cambridge, Mass.).

Hornfeck, Bernhard. Zur Struktur gewisser Primzahlsätze. J. Reine Angew. Math. 196 (1956), 156–169.

The author considers what properties of the sequence of prime numbers and other sequences are used in various density theorems, and, in the light of these results, generalizes them. Let T denote the set of all sequences $\mathfrak{A}=\{a_1, a_2, \cdots\}$ of positive integers such that $(a_i, a_j)=1$ for all $i\neq j$. It is shown that, if $\mathfrak{A}\in T$, d is a positive integer and $A(x; 0, d)$ is the number of $a_i\in\mathfrak{A}$ for which $a_i\leqq x$ and $a_i+d\in\mathfrak{A}$, then, uniformly in d,

$$A(x;0, d)<c_1\frac{x}{\log^2 x}\sum_{q|d}\frac{1}{q}\ (q \text{ square free}).$$

This result is used to show that $\delta^*\{\mathfrak{A}+f(\mathfrak{B})\}>0$, where $\mathfrak{A}\in T$ and has density lying between positive multiples of $x/\log x$, and where $\mathfrak{B}$ has positive density. Here δ^* denotes the upper asymptotic density and $f(\mathfrak{B})$ is the sequence $(f(b_1), f(b_2), \cdots)$, where $b_i\in\mathfrak{B}$ and f is a polynomial with integral coefficients and positive highest coefficient. This generalizes a result of Romanov, which is got by taking $\mathfrak{A}$ to be the sequence of primes, $\mathfrak{B}$ to be the sequence of non-negative integers and $f(x)$ to be x^m. Other results of Romanov, Prachar and Kai-Lai Chung are generalized. For example, it is shown that if $\mathfrak{A}$ satisfies the conditions just stated, a is an integer greater than unity, $\mathfrak{G}^*$ is a subset of $\{1, a, a^2, a^3, \cdots\}$, and $\mathfrak{H}^*=\mathfrak{A}+\mathfrak{G}^*$, then $H^*(x)>cxG^*(x)/\log x$, for sufficiently large x, where $H^*(x)$ is the number of members of $\mathfrak{H}^*$ which do not exceed x, and $G^*(x)$ is defined similarly. *R. A. Rankin.*

See also: Dwork, p. 556; Ankeny, Brauer and Chowla, p. 565.

Theory of Algebraic Numbers

★ **Kubota, Tomio. Density in a family of abelian extensions.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 77–91. Science Council of Japan, Tokyo, 1956.

Let A be a finite abelian group of order n. Let k be a finite algebraic number field, Ω the algebraic closure of k and F the family of all abelian extensions K/k in Ω such that the Galois group of K/k is isomorphic to A. The absolute norm of the conductor $\mathfrak{f}_K$ of such an extension K/k is denoted by $N(\mathfrak{f}_K)$. For any subfamily F_1 of F, it is proved that the sum $\sum N(\mathfrak{f}_K)^{-s}$ taken over all K in F_1 converges for $s>1$ and defines a function $f(s; F_1)$. If F_2 is a subfamily of F_1, the limit of $f(s; F_2)f(s; F_1)^{-1}$ for $s\to1$ $(s>1)$ is then called the density of F_2 in F_1 and is denoted by $\omega(F_2; F_1)$. The properties of $f(s; F_1)$ and $\omega(F_2; F_1)$ are studied for various subfamilies F_1, F_2 of F, and, among others, the following theorem is proved: Let $\mathfrak{m}$ be an ideal of k and let $\mathfrak{p}$ be a prime ideal of k dividing $\mathfrak{m}$ and prime to 2. Let F_1 be the subfamily of all K in F such that $\mathfrak{f}_K$ is prime to $\mathfrak{m}$ and F_2 the subfamily of those K in F_1 such that $\mathfrak{p}$ is completely decomposed in K. Then, $\omega(F_2; F_1)=1/n$. *K. Iwasawa* (Cambridge, Mass.).

★ **Artin, Emil. Representatives of the connected component of the idèle class group.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 51–54. Science Council of Japan, Tokyo, 1956.

A direct description is given of the connected component of the identity in the idèle class group of an algebraic number field. Let r_1 be the number of real infinite primes, r_2 the number of complex infinite primes of the algebraic number field K, and let $r=r_1+r_2-1$. Let R denote the additive group of the real numbers, Z the additive group of the integers, Z^+ the completion of Z with respect to the topology in which the ideals of Z constitute a fundamental system of neighborhoods of 0. Imbed Z in the direct sum $R+Z^+$ by the diagonal map, and let $S=(R+Z^+)/Z$. Then S is compact, connected, and infinitely and uniquely divisible. It is shown that the connected component of the idèle class group of K is isomorphic with the direct product of R, r copies of S, and r_2 copies of R/Z.

This isomorphism is obtained quite explicitly as follows: let U be the group of idèles of product formula value 1 whose components at the finite primes are units and whose components at the real infinite primes are positive. The ordinary integral exponentiation in U is extended to an exponentiation with exponents in $R+Z^+$. For a real number t, let $f_j(t)$ be the idèle whose component at the jth complex infinite prime is $e^{2\pi it}$ and all whose other components are 1. Let $e_1, \cdots, e_r$ be independent totally positive units of K. It is shown that an idèle of the form $e_1^{v_1}\cdots e_r^{v_r}f_1(t_1)\cdots f_{r_2}(t_{r_2})$, with $v_k\in R+Z^+$, is a principal idèle if and only if each v_k and each t_j is an ordinary integer, and that the connected component of the idèle class group is the direct product of R and the image of the group of the idèles of the above form.
G. P. Hochschild (Princeton, N.J.).

Ankeny, N. C.; Brauer, R.; and Chowla, S. A note on the class-numbers of algebraic number fields. Amer. J. Math. 78 (1956), 51–61.

It was shown by Landau [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1918**, 79–97] that the class number $h(F)$ of the algebraic number field F (of degree n) satisfies $h(F) < c|d(F)|^{\frac{1}{2}}|\log d(F)|^{n-1}$, where $d(F)$ is the discriminant, and c depends on n only. The present authors show that Landau's estimate is quite sharp. They prove (i) if ε, r_1, r_2, n are given ($\varepsilon > 0$, $n \geqq 2$, $r_1 + 2r_2 = n$, r_1 and r_2 are integers > 0) then there exist infinitely many fields F, having r_1 real and $2r_2$ imaginary conjugate fields, with

$$h(F) > |d(F)|^{\frac{1}{2}-\varepsilon}.$$

The totally real case ($r_2 = 0$) is obtained from equations of the type $f_n(x) = (x-a_1)\cdots(x-a_{n-1})(x-N)+1 = 0$, where $a_1, \cdots, a_{n-1}$ are distinct fixed integers, and N is a sufficiently large integer. The following lemma is applied to the discriminant $g(N)$ of $f_N(x)$: (ii) Let $g(x)$ be an integer-valued polynomial of degree s, such that the g.c.d. of the set $\{g(k)|k=0, \pm 1, \pm 2, \cdots\}$ is 1. Let $\rho \geqq 0$ be given, and let U be sufficiently large. Then there are at least $cU(\log\log U)^{-s}$ numbers N in the interval $(U, 2U)$ such that $g(N)$ has no prime factors $< (\log U)^{\rho}$; c does not depend on U.

The authors use (ii) for showing that

$$\text{(iii)} \qquad |d(F_n)| > (\log N)^{\rho}$$

for infinitely many values of N (ρ and n are given; F_n is the field generated by the largest root of the equation $f_n(x) = 0$). Now (i) follows by a theorem of R. Brauer [Amer. J. Math. **69** (1947), 243–250; MR **8**, 567] to the effect that $h(F)R(F) > |d(F)|^{\frac{1}{2}-\delta}$ (n and $\delta > 0$ given, $|d(F)|$ sufficiently large; $R(F)$ is the regulator of F). The general case $r_2 > 0$ is dealt with analogously.

N. G. de Bruijn (Amsterdam).

Godwin, H. J. Real quartic fields with small discriminant. J. London Math. Soc. 31 (1956), 478–485.

A method is given for determining the totally real algebraic fields of degree 4 with discriminant Δ less than an assigned bound, and a table is given of all such fields with $\Delta < 11664$. This supplements the table given by Delone and Faddeev [Trudy Mat. Inst. Steklov. **11** (1940); MR **2**, 349] for $\Delta \leqq 8112$. The method is based on simple considerations from the geometry of numbers, followed by a detailed analysis of cases. The author points out that there are two distinct fields of discriminant 16448, one generated by $\sqrt{(17+4\sqrt{2})}$ and the other by $x^4 - 10x^3 + 30x^2 - 32x + 10 = 0$. *H. Davenport.*

Godwin, H. J. On totally complex quartic fields with small discriminants. Proc. Cambridge Philos. Soc. 53 (1957), 1–4.

A general theorem is proved on algebraic number fields of degree n having no (non-trivial) subfield. Applied to totally complex quartic fields K, along with some elementary considerations, it implies the existence of a generating polynomial for K with coefficients bounded relative to the discriminant D of K. Thus, all K's with $D \leqq \Delta$ are given by a set of polynomials, tabulated by means of the bounds. The D's of corresponding fields can be evaluated, and the identity or non-identity of two fields with the same D can be determined by the methods of the paper reviewed above. For $\Delta = 1458$, there are exactly 17 D's, and fields of the same D are identical. Each of the 17 D's is given in a table which also exhibits a polynomial with a root θ generating the field, such that $1, \theta, \theta^2, \theta^3$ is a basis. The author also tabulates earlier results of Delone and Faddeev [Trudy Mat. Inst. Steklov. **11** (1940); MR **2**, 349] on totally complex quartic fields with small discriminants having a quadratic subfield.

R. Hull (Los Angeles, Calif.).

Mahler, K. A remark on Siegel's theorem on algebraic curves. Mathematika 2 (1955), 116–127.

Siegel has proved that an irreducible curve $f(x, y) = 0$, of genus $\geqq 1$, where f is a polynomial with algebraic coefficients, has only a finite number of points (x, y) satisfying the following conditions: y is in some preassigned algebraic number field K, and, for some fixed integer $j > 0$, jx is an integer in K. This theorem is generalized as follows: let $f(x, y) = 0$ be an arbitrary irreducible algebraic curve of genus $\geqq 1$ (the coefficients of f being complex numbers); let X be a finitely generated additive subgroup of the field C of complex numbers, and let Y be a vector subspace of finite dimension of C (considered as a vector space over the field of rationals); then $f(x, y) = 0$ has only a finite number of points such that $x \in X$, $y \in Y$. The proof consists in reducing this statement to the statement of Siegel's theorem by a suitable specialisation of the coefficients of f and of the generators of X and Y.

C. Chevalley (Paris).

Reiner, Irving; and Swift, J. D. Congruence subgroups of matrix groups. Pacific J. Math. 6 (1956), 529–540.

The author proves the following two theorems.

1. Let G be the group of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with elements a, b, c, d in the ring $\mathfrak{D}$ of integers in an algebraic number field. For any ideal $\mathfrak{N}$ in $\mathfrak{D}$ let $G(\mathfrak{N})$ denote the subgroup of $\mathfrak{D}$ defined by $c \equiv 0 \pmod{\mathfrak{N}}$. Then if H is a subgroup of G satisfying $G(\mathfrak{M}\mathfrak{N}) \subset H \supset G(\mathfrak{N})$ and $\mathfrak{M}$ is prime to 6 then $H = G(\mathfrak{D}\mathfrak{N})$ for some $\mathfrak{D} \supset \mathfrak{M}$ [for the special case of the ordinary modular group cf. Newman, Duke Math. J. **22** (1955), 25–32; MR **16**, 801].

2. Let r be a rational integer $\geqq 2$ and let M_r be the group of all $r\times r$ matrices $(a_{i,j})$ $(i, j = 1, 2, \cdots, r)$, where the a_{ij} are rational integers. If m and n are positive integers let C_m and R_n be the subgroups of M_r defined by $a_{i1} \equiv 0 \pmod{m}$ $(i = 2, 3, \cdots, r)$ and $a_{rj} \equiv 0 \pmod{n}$ $(j = 1, 2, \cdots, r-1)$ respectively. Then if H is a subgroup of M_r satisfying

$$(C_{am} \cap R_{bn}) \subset H \subset (C_m \cap R_n)$$

and a, b, m, n are integers such that $(am, bn) = 1$, then $H = C_{\alpha m} \cap R_{\beta n}$ where $\alpha|a$, $\beta|b$. *H. D. Kloosterman.*

See also: Ramanathan, p. 557; Tanaka, p. 563; Taniyama, p. 601; Deuring, p. 601.

Geometry of Numbers

Pipping, Nils. Approximation zweier reellen Zahlen durch rationale Zahlen mit gemeinsamem Nenner. Acta Acad. Abo. 21 (1957), no. 1, 17 pp.

The author studies the problem to find algorithms containing a sequence of integer-valued systems of numbers $T^{(\nu)} = \{T_i^{(\nu)}; i = 1, 2, 3\}$, $\nu = 1, 2, \cdots$ with the property that this sequence will contain only relatively few elements but if possible all systems T with the property

$$\Delta_{j\nu}(T) = \min_{\{\max T_i \leqq T_\nu\}} \max_i \left| u_j\left(\frac{T_j}{u_j} - \frac{T_i}{u_i}\right)\right|$$

where $u=\{u_1>u_2>u_3>0\}$ is a set of real numbers.

He investigates by studying some numerical examples the effectiveness of the algorithms of Törnqvist, Jacobi, Brun and Pipping for this purpose. The algorithm of Törnqvist gives often such especially good diophantine approximations T_i/T_j to the numbers u_i/u_j but does not contain all such systems of integers. The branching algorithm of Pipping seems to give them all. Pipping tries to define an algorithm for which

$$\max_{i,j}\left|u_j\left(\frac{T_j}{u_j}-\frac{T_i}{u_i}\right)\right|$$

is a decreasing sequence of T_n containing all especially good systems (T_1, T_2, T_3) but he is not able to prove that his method always will be successful. In the numerical examples considered this is however the case.

L. Tornqvist (Helsinki).

Davenport, H. Note on irregularities of distribution. Mathematika 3 (1956), 131–135.

Roth [Mathematika 1 (1954), 73–79; MR 16, 575] has shown that there exists an absolute positive constant c with the following property: Let (x_n, y_n) $(1\leqq n\leqq N)$ be N points in the unit square $0\leqq x<1$, $0\leqq y<1$; and for $0\leqq\xi<1$, $0\leqq\eta<1$ let $S(\xi, \eta)$ denote the number of these in $0\leqq x_n<\xi$, $0\leqq y_n<\eta$. Then

$$\int_0^1\int_0^1 (S(\xi, \eta)-N\xi\eta)^2 d\xi d\eta>c \log N.$$

Davenport shows that c cannot be replaced by any function of N tending to infinity with N. The counterexample makes use of the fractional parts of a real number with bounded partial quotients. It is shown that the corresponding result of Roth in three dimensions would similarly be the best possible if a problem of Littlewood about the existence of a pair of irrationals with certain properties has a positive solution. *J. W. S. Cassels.*

Popken, J. Un théorème sur les nombres transcendants. Bull. Soc. Math. Belg. 7 (1955), 124–130.

Let $F(z)$ be a polynomial with algebraic coefficients, which vanishes at zero but not identically. Let r be a rational number, and for $n=1, 2, \cdots$ put

$$F_n=\begin{cases}\dfrac{1}{n+r}\left(\dfrac{d^n}{dz^n}e^{(n+r)F(z)}\right)_{z=0} & \text{if } n+r\neq 0,\\ 0 & \text{if } n+r=0.\end{cases}$$

Then the series

$$f(z)=\sum_{n=1}^{\infty} F_n\frac{z^n}{n!}$$

defines an analytic function f. It is shown that for each algebraic number z different from the zeros of F, the function f is regular, and $f^{(k)}(z)$ is transcendental for $k=0, 1, 2, \cdots$. This theorem generalizes a result of the reviewer [Proc. Amer. Math. Soc. 2 (1951), 401–403; MR 13, 16] dealing with the special case $F(z)=z$.

W. J. LeVeque (Ann Arbor, Mich.).

See also: Ballantine, p. 561; van der Waerden, p. 562; Bloh, p. 595.

ANALYSIS

Functions of Real Variables

Goldberg, Richard R. Pseudo-multiplicative functions. Math. Mag. 30 (1957), 145–148.

Elementary closure and rate-of-growth properties for non-decreasing, non-negative, real-valued functions f for which

$$f(xy)\geqq f(x)f(y)$$

for all x, y in the domain of f, this domain being a multiplicative semi-group of non-negative reals.

T. A. Botts (Charlottesville, Va.).

Brunk, H. D.; Ewing, G. M.; and Utz, W. R. Some Helly theorems for monotone functions. Proc. Amer. Math. Soc. 7 (1956), 776–783.

Let S denote a closed n-dimensional interval of the Euclidean n-space. A function $F(x_1, x_2, \cdots, x_n)$ defined on S is called "monotone" by the authors if F is monotone nondecreasing in each x_i $(i=1, 2, \cdots, n)$ and if the first and second differences of F are non-negative. The following theorem [of a type which was first treated by E. Helly, Akad. Wiss. Wien. S.-B. IIa. 121 (1912), 265–297] is proved by the authors: Let $\{F_q(x_1, x_2, \cdots, x_n)\}$ be a sequence of monotone functions on S with

$$|F_q(x_1, x_2, \cdots, x_n)|\leqq A \quad (q=0, 1, 2, \cdots).$$

There exists a sequence of integers $q_0<q_1<q_2<\cdots$ and a monotone function $F(x_1, x_2, \cdots, x_n)$ bounded by $\pm A$ such that

$$\lim_{i\to\infty} F_{q_i}(x_1, x_2, \cdots, x_n)=F(x_1, x_2, \cdots, x_n) \text{ on } S.$$

For the proof of this result the following theorem is used: If $F(x_1, x_2, \cdots, x_n)$ is a monotone function on S, then there exists a set R of points of countably many $(n-1)$-spaces $x_i=\alpha_i^j$ $(i=1, 2, \cdots, n;\ j=1, 2, \cdots)$ such that F is continuous at all points of $S-R$. As consequences of both these results, also theorems on extended real valued functions F (i.e. $-\infty\leqq F(x_1, x_2, \cdots, x_n)\leqq+\infty$) are stated. Finally, by means of a simple example in two variables, it is shown that the first mentioned theorem does not hold if the functions F_q are only monotone in each of their variables. (Remark: Another generalization of Helly's theorems has recently been given by O. M. Nikodým [Rend. Sem. Mat. Univ. Padova 24 (1955), 265–286; MR 17, 594]; there the assumptions are stronger, namely for functions of n variables the differences up to the nth ones are supposed to be non-negative.) *A. Rosenthal.*

Mikolás, Miklós. Construction des familles de fonctions partout continues non dérivables. Acta Sci. Math. Szeged 17 (1956), 49–62.

The author derives new sufficient conditions under which $f(x)=\sum_{k=0}^{\infty} c_k\varphi(\nu_k(x))$ is continuous and nowhere differentiable (c.n.d.). These conditions are of the following type. $\varphi(x)$ is convex (or concave) on segments, or $\varphi(x)$ satisfies a Lipschitz condition; the ν_k are integers and the c_k and ν_k satisfy certain limit conditions. The theorems contain most of the known examples as special cases and improve many known results. For instance, the author finds that Weierstrass' function $\sum c^k \cos \nu^k x$ is c.n.d. under the conditions $\nu>1$, $\nu c\geqq 1+\frac{1}{2}\pi$, which are weaker than the known conditions.

A. Heyting (Amsterdam).

See also: Eyraud, p. 551; Stein, p. 575; Feller, p. 575; Pi Calleja, p. 587; Kapuano, p. 589.

Measure, Integration

★**McShane, Edward James. Integration.** Princeton University Press, Princeton, N. J., 1944. (Fourth printing 1957). viii+394 pp. $2.95.
This book was reviewed in MR 6, 43.

Potts, D. H. Elementary integrals. Amer. Math. Monthly 63 (1956), 545–554.

The function $y=y(x)$ is an algebraic function of x if it is defined by an irreducible relation

$$a_0y^n+a_1y^{n-1}+\cdots+a_n=0,$$

where $a_0, a_1, \cdots, a_n$ are polynomials in x and $a_0\not\equiv 0$. It is to be understood that $y=f(x)$ where f is constructed from the three basic operations, algebraic, exponential, logarithmic. Algebraic implies the extraction of roots and the usual rational operations. Ostrowski's method of field extension [Ritt, Integration in finite terms, Columbia Univ. Press, 1948; MR 9, 573] is used. If F is a differential field (F contains all constants; α, $\beta \in F$ implies $\alpha+\beta$, $\alpha\cdot\beta \in F$; $\alpha \in F$, $\alpha' \in F$; $\alpha \in F$, $1/\alpha \in F$ unless $\alpha\equiv 0$) and θ a piecewise differentiable function not in F then the class of rational functions of θ with coefficients in F is a field. If it also is a differential field it is a simple extension of F. A simple extension of $F(\theta_1, \cdots, \theta_{n-1})$ is denoted by $F(\theta_1, \cdots, \theta_n)$ and is called an extension of F. Eight theorems are proved, two of which are:

Theorem 3. α, $e^{n\alpha} \in F$; $e^{p\alpha} \notin F$, $p=0, 1, \cdots n-1$; $\beta_0, \cdots, \beta_{n-1} \in F$; $\theta=e^\alpha$. Then there exists $\gamma_0, \cdots, \gamma_{n-1}$ not all zero, $\in F$ such that

$$\left[\sum_{k=0}^{n-1}\beta_k\theta^k\right]\left[\sum_{k=0}^{n-1}\gamma_k\theta^k\right] \in F.$$

Theorem 8. α, $\beta \in F$, $e^{n\alpha} \notin F$, $n=1, 2, \cdots$, $\gamma=e^\alpha\beta$ is elementary with respect to F, i.e. γ is contained in an extension of F. Then there exists $\delta \in F$ such that $\gamma=e^\alpha\delta+\text{constant}$. *R. L. Jeffery* (Kingston, Ont.).

Il'in, V. P. Generalization of an integral inequality. Uspehi Mat. Nauk (N.S.) 11 (1956), no. 4(70), 131–138. (Russian)

The author generalizes the inequality of Hilbert-Riesz, which is the case $n=m=1$ of the following: Let p, q be real numbers both >1 such that $1/p+1/q>1$, let $m\leq n$ be integers, let X be an n-dimensional Euclidean space and let Y be an m-dimensional subspace of X, let p' and q' be the conjugate indices of p and q (i.e., $1/p+1/p'=1$); if $x \in X$ and $y \in Y$, let $r=|x-y|$, the Euclidean distance between x and y, and let $\lambda=n/p'+m/q'$. Then there exists a constant $K=K(m, n, p, q)$ such that for all f in $L^p(X)$ and g in $L^q(Y)$

$$\int_{X\times Y}\frac{f(x)g(y)}{r^\lambda}\,dxdy\leq K\|f\|_{L^p(X)}\|g\|_{L^q(Y)}.$$

An upper bound for the value of K is given. *M. M. Day.*

Kunugi, Kinjirô. Application de la méthode des espaces rangés à la théorie de l'intégration. I. Proc. Japan Acad. 32 (1956), 215–220.

Soit ε l'ensemble des fonctions $f(x)$ réelles finies en escalier, définies sur un intervalle donné $[a, b]$ $(a<b)$ réel borné et fermé. La function $f(x)$ est dite en escalier si $[a, b]$ est réunion d'un nombre fini d'intervalles dans l'intérieur de chacun desquels $f(x)$ est constante. L'A., au moyen de définitions convenables et en utilisant ses travaux antérieurs [mêmes Proc. 30 (1954), 553–556, 912–916; MR 17, 389], considère ε comme un espace uniforme rangé où il définit des suites fondamentales de voisinages. Il utilise ces moyens pour donner, pour des fonctions $F(x)$ réelles, une définition de l'intégrale $\int_a^b F(x)dx$, définition plus longue que celle de Lebesgue. *A. Appert* (Angers).

Pucci, Carlo. Alcune proprietà degli involucri. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 20 (1956), 294–298.

For a set X in E^n denote, if $\rho>0$, by X_ρ the set of those points with distance less than ρ from X, and, if $\rho<0$, the set of those points whose distance from the complement of X is less than $-\rho$. By $-r(X)$ we denote the greatest lower bound of the (non-positive) ρ for which X_ρ is non-empty. If ∂X is the boundary of X, then

$$\mu_s\partial X=\lim_{\rho\to 0+}\frac{(\partial X)_\rho}{2\rho},\ \mu_i\partial X=\lim_{\rho\to 0-}\frac{(\partial X)_\rho}{2\rho}$$

are the upper and lower Minkowski measures of ∂X. If they are equal we denote them by $\mu\partial X$. It is proved that $\mu\partial(X_\rho)$ exists for almost all $\rho\geqq -r(X)$ and that for $|\partial X|=0$, where $|\ |$ indicates Lebesgue measure,

$$|X|=\int_{-r(X)}^0\mu\partial(X_\rho)d\rho.$$

H. Busemann (Los Angeles, Calif.).

Functions of Complex Variables

Bögel, Karl. Über die Cauchy-Riemannschen Differentialgleichungen. Math. Nachr. 15 (1956), 87–88.

The author shows that if the real and imaginary parts of the function $f(z)$ satisfy the Cauchy-Riemann differential equations in a region B, then the set of points of the region at which $f'(z)$ fails to exist is nowhere dense in B. *P. Civin* (Eugene, Oreg.).

Ikoma, Kazuo. On Ahlfors' discs theorem and its application. Tôhoku Math. J. (2) 8 (1956), 101–107.

The reviewer's theorem [Acta Soc. Sci. Fenn. (N.S.) 2 (1933), no. 2] is extended to quasiconformal mappings ("pseudo-analytic" in the Japanese terminology) and a corresponding version of Bloch's theorem is proved. The results are tied to the condition $\lim_{z\to 0}|f(z)|/|z|^{1/K}>0$ which is a natural restriction for the method, but not for the problem. *L. Ahlfors* (Cambridge, Mass.).

Mel'nik, I. M. Limit values of an analytic function represented by a curvilinear integral. Soobšč. Akad. Nauk Gruzin. SSR 17 (1956), 681–688. (Russian)

The author considers the integral

$$F(z)=\int_L\frac{P(t, z)}{Q(t, z)}\,dz,$$

where P and Q are entire functions of z for every t on the smooth curve L. He imposes conditions on Q which ensure that the equation $Q(t, z(t))=0$, $t \in L$, defines a set of non-intersecting, bounded curves $\{\Gamma\}$ in the z-plane. He investigates the behavior of $F(z)$ as z approaches a point $w \in \Gamma$. By imposing a Hölder condition (in t) on P the problem is reduced to the known case of an integral of

Cauchy-type, F attains boundary values F^+ and F^- as z approaches from the left or from the right of Γ. Their values can be written down explicitly. The author also gives sufficient conditions under which the formula

$$\int_L \frac{d\tau}{Q(\tau, w)} \int_L \frac{P(\tau, \sigma)}{R(\tau, \sigma)} d\sigma = -\pi^2 \frac{P(t, t)}{Q_\tau'(t, w) R_\tau'(t, t)} + \int_L d\sigma \int_L \frac{P(\tau, \sigma) d\tau}{Q(\tau, w) R(\tau, \sigma)} \quad (w = z(t) \in \Gamma)$$

olds . *W. H. J. Fuchs.*

Jenkins, James A. Some new canonical mappings for multiply-connected domains. Ann. of Math. (2) **65** (1957), 179–196.

Using a continuity method the author derives some new canonical mappings of multiply-connected regions. Typical is the following: a region with boundaries $C_1, \cdots, C_n$ can be mapped, essentially uniquely, so that C_1 corresponds to an outer rectangle R_1 and each C_i $(i>1)$ correspond to an inner rectangle R_i with given ratio of the sides. The proof uses quasiconformal mappings and leans on Teichmüller's fundamental theorem. A more general setting is given in which the boundary lines are trajectories of a quadratic differential. The author promises to return with applications. *L. Ahlfors.*

★**Stoïlow, S. Leçons sur les principes topologiques de la théorie des fonctions analytiques. Deuxième édition, augmentée de notes sur les fonctions analytiques et leurs surfaces de Riemann.** Gauthier-Villars, Paris, 1956. xvi+194 pp. 1700 francs.

This is a photographic reproduction of the first edition of 1938. The four notes that have been appended are reprints of later articles of the author. *L. Ahlfors.*

Wirth, Eva Maria. Über die Bestimmung des Typus einer Riemannschen Fläche. Comment. Math. Helv. **31** (1956), 90–107.

Let $f(x)$ denote a monotone increasing continuous piecewise analytic real-valued function on $\{x \geqq 0\}$, satisfying $f(0)=0$. Let Ω denote the doubly-connected Riemann surface obtained from the strip $\{x>0;\ 0 \leqq y \leqq 1\}$ by identifying the points x and $x+f(x)+i$, $x>0$. Ω is conformally equivalent to an annulus $\{1<|z|<R(\leqq +\infty)\}$. Let ω denote the mapping of the strip onto the annulus which satisfies $\omega(0)=\omega(i)=1$ and defines a univalent conformal map of Ω onto the annulus. For $1 \leqq \rho < R$ an index $n(\rho)$ is introduced which counts the essential number of intersections (in a sense made precise in the paper) of the image of the ray $\{x>0;\ y=0\}$ with respect to ω.

Theorem: If $\int^\infty (1+f^2)^{-1} dx < \infty$ and $n(\rho)=O(1)$, then $R<\infty$. It is shown that this theorem cannot be concluded from a type criterion given by Volkovyskii. *M. Heins.*

Akutowicz, Edwin J. A qualitative characterization of Blaschke products in a half-plane. Amer. J. Math. **78** (1956), 677–684.

Let $b(z)$ be a Blaschke product for the upper half plane. The author shows that

$$\int_{-\infty}^{\infty} (1+x^2)^{-1} \log |b(z)| dx \to 0$$

as $y \to 0+$. Conversely let $F(z)$ be regular for $y>0$, $|F(z)|<1$, and $\int_{-\infty}^{\infty} (1+x^2)^{-1} \log |F(z)| dx \to 0$ as $y \to 0+$. Then

$$F(z) = e^{ikz+ic} b(z),$$

where $k(\geqq 0)$ and c are real numbers, and $b(z)$ is the Blaschke product over the zeros of F in the upper half plane. *R. P. Boas, Jr.* (Evanston, Ill.).

Sunyer Balaguer, F. Asymptotic values of entire functions. Collect. Math. **8** (1955–1956), 187–211. (Spanish)

The first part of this paper contains the detailed proofs of results announced previously [C. R. Acad. Sci. Paris **237** (1953), 548–550; MR **15**, 207]: for an entire function of finite order the number of asymptotic values can be estimated more closely than in the Denjoy-Carleman-Ahlfors theorem if something is said about how fast they are approached. In the second part similar results are obtained for functions of infinite order.

R. P. Boas, Jr. (Evanston, Ill.).

Robinson, R. M. A curious trigonometric identity. Amer. Math. Monthly **64** (1957), 83–85.

The functions z, $\sin z$ and $\sinh z$ are essentially the only functions $f(z)$ with the property that

$$|f(x+iy)| = |f(x)+f(iy)|.$$

Wasow, Wolfgang. Asymptotic development of the solution of Dirichlet's problem at analytic corners. Duke Math. J. **24** (1957), 47–56.

Let $u(x, y)$ be harmonic in a simply connected region whose boundary contains two analytic arcs L_1, L_2 with common endpoint P at which they form an interior angle $\alpha\pi$ with $0<\alpha \leqq 2$, and suppose u is continuous on L_1+L_2 and holomorphic on L_1, L_2, resp. Adapting techniques of H. Lewy [Univ. California Publ. Math. (N.S.) **1** (1950), 247–280; MR **12**, 691] and R. S. Lehman [Comm. Pure Appl. Math. **7** (1954), 393–439; MR **16**, 296] the author obtains asymptotic expansions for u in the neighborhood of P. In particular, if r is the distance from P and m is an integer, $u=u_1+u_2$ where, as $r\to 0$, all partials of u_1 are $O(1)$, and $u_2=O(r^m \log r)$ or $O(r^{1/\alpha})$ according as $\alpha=1/m$ or not; moreover the latter relations may be formally differentiated indefinitely. *H. A. Antosiewicz.*

Zamorski, J. Equations satisfied by the extremal schlicht functions with a pole. Ann. Polon. Math. **3** (1956), 41–45.

Let Σ denote the class of functions

$$F(z) = \frac{1}{z} + b_1 z + b_2 z^2 + \cdots,$$

regular and schlicht for $0<|z|<1$. Let V_p denote the pth coefficient region for this class of functions, the function F corresponding to the point of V_p with coordinates $(x_1, y_1, x_2, y_2, \cdots, x_p, y_p)$, where $b_k=x_k+iy_k$. V_p contains the origin and is connected and closed. Let

$$E(x_1, y_1, x_2, y_2, \cdots, x_p, y_p)$$

be an arbitrary real-valued function, continuous together with its partial derivatives and satisfying

$$|\mathrm{grad}\, E|^2 = \sum \left(\left(\frac{\partial E}{\partial x_k} \right)^2 + \left(\frac{\partial E}{\partial y_k} \right)^2 \right) > 0$$

throughout V_p. Let $E_k = \frac{1}{2}(\partial E/\partial x_k - i\partial E/\partial y_k)$, $\bar{E}_k$=conjugate of E_k. E has its extreme value on the boundary of V_p.

The author's main theorem is that the function $F(z)$, regular and schlicht for $0<|z|<1$, whose coefficients give the extreme value of the function E, satisfies the equation

$$(1)\qquad \left(\frac{zF'(z)}{F(z)}\right)^2 \sum_{k=2}^{p+1} A_k F^k(z) = -\sum_{k=-p-1}^{p+1} \frac{B_k}{z^k}.$$

Formulae for each A_k and B_k are given in terms of the coefficients b_k by means of the function E.

The proof of the author's theorem is based on analogous results of Schaeffer and Spencer [Coefficient regions for schlicht functions, Amer. Math. Soc. Colloq. Publ., v. 35 New York, 1950; MR 12, 326] for regular functions, with slight modifications to take care of the pole. {The reviewer wishes to point out that the variational formulae for the coefficients b_k obtained by the author were previously found by G. Springer [Trans. Amer. Math. Soc. 70 (1951), 421–450; MR 13, 24]. Also it should be noted that while the complete details of the author's theorem may not have been previously published, they appear to have been known and used by Jenkins [Proc. Amer. Math. Soc. 4 (1953), 595–599; MR 15, 114].}

It has been conjectured that the pth coefficient b_p of $F(z)$ has a maximum modulus obtained by the schlicht function

$$\phi_p = \{z^{(p+1)/2} + z^{-(p+1)/2}\}^{2/(p+1)} = \frac{1}{z} + \frac{2}{p+1} z^p + \cdots$$

[see W. Wolibner, Studia Math. 11 (1949), 126–132; MR 12, 16], and the conjecture has been verified for $p=1$ and 2 [Golusin, Mat. Sb. N.S. 3(45) (1938), 321–330; Schiffer, Bull. Soc. Math. France 66 (1938), 48–55]. The author verifies that the function $\phi_p(z)$ satisfies the appropriate differential functional equation (1), thus providing support for the conjecture that $|b_p| \leq 2/(p+1)$. {Unfortunately, the author apparently was not aware that the conjecture has recently been shown to be false by Garabedian and Schiffer [Ann. of Math. (2) 61 (1955), 116–136; MR 16, 579] who obtained the sharp bound $|b_3| \leq \frac{1}{2} + e^{-6}$ from an appropriate form of the author's equation (1).} *M. S. Robertson* (New Brunswick, N.J.).

Walsh, J. L. Note on degree of approximation to analytic functions by rational functions with preassigned poles. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 927–930.

Several theorems are proved relating to the degree of approximation to analytic functions f by rational functions with preassigned poles. Open sets and closed sets are considered whose boundary J consists of a finite number of disjoint analytic Jordan curves and $f^{(p)}(z) \in L(p, \alpha)$ for $z \in J$. *P. Davis* (Washington, D.C.).

Ronkin, L. I. On types of an entire function of two complex variables. Mat. Sb. N.S. 39(81) (1956), 253–266. (Russian)

Entire functions $f(z, u)$ are considered for which there exist constants σ_1 and σ_2 and an A_ε for any $\varepsilon>0$ such that $|f(z,u)| \leq A_\varepsilon \exp[(\sigma_1+\varepsilon)|z|+(\sigma_2+\varepsilon)|u|]$. Moreover, if σ_1' and σ'_2 satisfy either $\sigma_1' \leq \sigma_1$, $\sigma_2' < \sigma_2$ or else $\sigma_1' < \sigma_1$, $\sigma_2' \leq \sigma_2$, then there is a sequence (z_n, u_n) such that $|z_n|+|u_n| \to \infty$ and $|f(z_n, u_n)| \geq \exp(\sigma_1'|z_n|+\sigma_2'|u_n|)$. With $f=\sum a_{nk} z^n u^k/(n!k!)$ the related functions $\sum a_{nk} z^{-n-1} u^{-k-1}$ and $\sum a_{nk} u^k/(k! z^{n+1})$ are considered and generalizations and extensions of results for entire functions of exponential type in one variable are obtained. *N. Levinson.*

See also: Parodi, p. 554; Artemiadis, p. 575; Morimoto, p. 583; Rudin, p. 578.

Geometrical Analysis

Frölicher, Alfred; and Nijenhuis, Albert. Theory of vector-valued differential forms. I. Derivations in the graded ring of differential forms. Nederl. Akad. Wetensch. Proc. Ser. A. 59=Indag. Math. 18 (1956), 338–359.

On appelle forme scalaire (resp. vectorielle) sur un A-module unitaire V, toute application multilinéaire à valeurs dans A (resp. dans V). Si ω et π sont des formes scalaires de degré p et q et si L et M sont des formes vectorielles de degré q et p, on définit les produits suivants

$$(\omega \wedge \pi)(u_1, \cdots, u_{p+q}) = (1/(p!q!))\textstyle\sum_\alpha \pm \omega(u_{\alpha_1} \cdots u_{\alpha_p})\pi(u_{\alpha_{p+1}}, \cdots, u_{\alpha_{p+q}})$$

$$(\omega \wedge L)(u_1, \cdots, u_{p+q}) = (1/p!q!))\textstyle\sum_\alpha \pm \omega(u_{\alpha_1} \cdots u_{\alpha_p})L(u_{\alpha_{p+1}}, \cdots, u_{\alpha_{p+q}})$$

$$(\omega \barwedge L)(u_1, \cdots, u_{p+q-1}) = (1/(p!(q-1)!))\textstyle\sum_\alpha \pm \omega(L(u_{\alpha_1}, \cdots, u_{\alpha_q}), u_{\alpha_{q+1}}, \cdots, u_{\alpha_{p+q-1}})$$

$$(M \barwedge L)(u_1, \cdots, u_{p+q-1}) = (1/(p!(q-1)!))\textstyle\sum_\alpha \pm M(L(u_{\alpha_1}, \cdots, u_{\alpha_q}), u_{\alpha_{q+1}}, \cdots, u_{\alpha_{p+q-1}}).$$

Les champs de vecteurs tangents (de classe infinie) à la variété X (de classe infinie) s'identifient aux dérivations de l'anneau F des fonctions réelles (de classe infinie) définies sur X_η; leur ensemble constitue un F-module T^1, à partir duquel on définit comme plus haut les modules T^{*q} et V^{*q} de formes scalaires et vectorielles. Les Auteurs établissent soigneusement la concordance entre les définitions globales et locales des notions précédentes.

La somme directe $T^* = \oplus_q T^{*q}$ est un anneau extérieur dont les dérivations sont également de caractère local. Chacune d'elles est définie entièrement par son action sur $T^{*0}=F$ et sur T^{*1}. Il n'existe pas de dérivation non triviale de degré $=-2$, et toutes celles de degré -1 s'annulent sur F.

Toute dérivation D de degré $=-1$ qui s'annule sur F est de la forme $D\omega = \omega \barwedge L$ où L est une forme vectorielle bien déterminée, et réciproquement, toute forme L définit de la sorte une dérivation (que l'on notera i_L) qui s'annule sur F (dérivation de type i_*).

La différentielle extérieure est une dérivation de T^* qui n'est pas de type i_*, mais bien de type d_*, en appelant dérivation de type d_*, toute dérivation D telle que $Dd=(-1)^r dD$, où r est le degré de D.

Toute dérivation D de type d_* est définie par son action sur F et correspond bijectivement à une forme vectorielle L telle que l'on ait

$$D\omega = [L, \omega] = d\omega \barwedge L + (-1)^r d(\omega \barwedge L) = (i_L d - (-1)^{r-1} d i_L)\omega.$$

En d'autres termes, on a $d_L = i_L d - (-1)^{r-1} d i_L$.

Toute dérivation de degré r est la somme d'une dérivation de degré r de type i_* et d'une dérivation de degré r de type d_*.

Le crochet $[D_1^*, D_2] = D_1 D_2 - (-1)^{r_1 r_2} D_2 D_1$ est une nouvelle dérivation de degré r_1+r_2 et, en particulier, le commutateur de deux dérivations de type d_* est une dérivation de type d_*, ce qui permet de définir le crochet $[L, M]$ par la relation $[d_L, d_M] = d_{[L,M]}$ .Les propriétés des concommitants scalaires et vectoriels $[L, \omega]$ et $[L, M]$ sont étudiées systématiquement par les auteurs qui appliquent leur théorie pour donner une démonstration nouvelle d'un théorème de Haantjes. *G. Papy.*

Functions with Particular Properties

Albertoni, Sergio. Sulla risoluzione del problema di Neumann per l'equazione $\Delta_2 u+ku=f$ in un dominio con punti angolosi. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. **90** (1956), 221–243.

Let D be a bounded plane region of finite connectivity having a boundary Γ with continuous curvature. Assume the function f satisfies a Hölder condition in D and that the function B is integrable and continuous almost everywhere on Γ. The author [same Rend. (3) **18(87)** (1954), 400–432; MR **17**, 474] has given an account of the Fredholm theory as it applies to the equation $\nabla^2 u-\lambda^2 u=f$ for the region D, where λ is a complex number and the normal derivative of u on Γ is assigned the given values B. In the present paper this Neumann problem is treated along the same lines for a connected region but now the boundary of the region is permitted to consist of a finite number of arcs each having continuous curvature. The boundary curve is not permitted to have cuspidal points.

F. G. Dressel (Durham, N.C.).

Fet, A. I.; and Bodrecova, L. B. Functions with simple level curves. Mat. Sb. **38(80)** (1956), 303–318. (Russian)

Consider a continuous real-valued function F with possible poles defined on an (open) domain D of the plane. In the author's terminology a point of D is called ordinary if it has arbitrarily small neighborhoods intersecting every level set, $F=$const, in a simple open arc, otherwise it is called a critical point. If the critical points of F in D are isolated, then F is said to have simple level curves in D. In general this property has no relation to differentiability of the function; however, if the function is analytic then its level curves are simple if the singularities are isolated. Three types of critical points are distinguished: (1) those at which the function takes the value $\pm\infty$ (poles), (2) the relative extrema, and (3) all those which are neither of these (they are called saddle points). The main result of the paper is to establish that if a function F has simple level curves in D then for every point p, $F(p)=c$, of D there is a neighborhood V_p together with a homeomorphism h_p of the unit disk of the xy-plane onto V_p carrying the origin onto p such that for the composite function $F_p'=F\circ h_p$ we have $F_p'=x+c$ if p is an ordinary point, or (1) $F_p'=\pm\ln(x^2+y^2)$, or (2) $F_p'=c\pm(x^2+y^2)$, or (3) $F_p'=c+\mathrm{Re}(x+iy)^n$, respectively according to the three types of singularities. Thus in particular it follows that a function F on D has as family of level curves a regular family, in the sense of W. Kaplan [Duke Math. J. **7** (1940), 154–195; MR **2**, 322] if and only if it has simple level curves and is without critical points. Moreover such a function is pseudo-harmonic in D, in the sense of M. Morse [Topological methods in the theory of functions of a complex variable, Princeton, 1947; MR **9**, 20] if and only if it has simple level curve and no relative extrema in D. These results are similar to those of Tôki [Osaka Math. J. **3** (1951), 101–122; MR **13**, 234].

W. M. Boothby (Evanston, Ill.).

Šabat, B. V. On mappings realizable as solutions of a Carleman system. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 3(69), 203–206. (Russian)

The author considers the elliptic system of first order partial differential equations

$$(1)\qquad u_x-v_y=au+bv,\quad u_y+v_x=cu+dv.$$

This system could also be written in the complex form

$$w_{\bar z}=Aw+B\bar w,$$

where $w=u+iv$, and A, B are complex functions. He remarks that while certain properties of analytic functions have been extended to solutions of (1) by Carleman, Vekua and the reviewer, solutions of (1) do not have the geometric properties of analytic functions. This is illustrated by constructing suitable counter-examples. {The reviewer remarks that if one associates with every solution w of (1) a function $\omega=\phi+i\psi$ so that $w=\phi F+\psi G$, where F, G are two particular solutions of (1), then the function ω is interior and quasi-conformal. For functions ω an extension of Riemann's metric theorem is known to hold. [For more precise statements and references see Bers, Bull. Amer. Math. Soc. **62** (1956), 291–331; MR **18**, 470].}

L. Bers (New York, N.Y.).

See also: Wasow, p. 568; Schäffer, p. 576; Landkof, p. 579; Shiga, p. 584; Florian, *p*. 595.

Special Functions

Al-Salam, W. A. On a characterization of some orthogonal functions. Amer. Math. Monthly **64** (1957), 29–32.

Put

$$\Delta_n(f)=f_n^2-f_{n+1}f_{n-1},\quad D_n(f)=f_n'^2-f_{n+1}'f_{n-1}'.$$

The reviewer [Bull. Calcutta Math. Soc. **46** (1954), 93–95; MR **16**, 694] showed that Nanjundiah's relation

$$(1-x^2)D_n(f)=N(n+1)\Delta_n(f)$$

characterizes the Legendre polynomials. The present paper contains the following results. 1. If $\{f_n\}$ is a sequence of polynomials such that $f_0=1$, $f_1=x$ and

$$D_n(f)=\Delta_n(f)+n(n-1)\Delta_{n-1}(f),$$

then $f_n=H_n(x)$, the Hermite polynomial. 2. If $f_0=1$, $f_1=x$ and $\Delta_n(f)=2^{-2(n-1)}(1-x^2)$, then $f_n=T_n(x)$, the Tchebycheff polynomial of the first kind. 3. If $f_0=1$, $f_1=2x$, $\Delta_n(f)=1$, then $f_n=U_n(x)$, the Tchebycheff polynomial of the second kind. 4. If $D_n(f)=x^2\Delta_{n-1}(f)$ and certain initial conditions are satisfied, then $f_n=x^nJ_n(x)$ or $x^nI_n(x)$.

L. Carlitz (Durham, N.C.).

Drazin, M. P. Another note on Hermite polynomials. Amer. Math. Monthly **64** (1957), 89–91.

The author shows that the identity of Nielsen-Dhar-Feldheim

$$H_m(x)H_n(x)=\sum_r 2^r r!\binom{m}{r}\binom{n}{r}H_{m+n-2r}(x),$$

where $H_n(x)$ is the Hermite polynomial of degree n, is a consequence of the obvious identity

$$(1+x^2)^v(1+x)^{m+n-2v}=\left\{1-\frac{2x}{(1+x)^2}\right\}^v(1+x)^{m+n}.$$

L. Carlitz (Durham, N.C.).

Dieulefait, Carlos E. On the zero sof the classical orthogonal polynomials in the asymptotic case. Rev. Un. Mat. Argentina 17 (1955), 25–27 (1956). (Spanish)

Details of formal calculations, for Jacobi polynomials, indicated by the author in a previous paper [Giorn. Ist.

Ital. Attuari **17** (1954), 36–46, especially § 6(c), p. 46; MR **17**, 962]. *H. P. Mulholland* (Birmingham).

Rodríguez Sanjuán, Antonio. Reduction of elliptic integrals to canonical forms in the real field. Rev. Acad. Ci. Madrid **50** (1956), 19–233. (Spanish)

The author claims, apparently with justice, that the standard treatments of elliptic integrals of the third kind involve complex transformations of real integrals and are consequently unsatisfactory, if not completely useless, for practical purposes. He presents an alternative treatment using only real transformations, which is if anything simpler than the usual treatment, and makes various other contributions. Let y denote the square root of a real quartic polynomial, and consider the integral $\int R(x, y)dx$, where R is a real rational function. The integral can be broken up into the integral of a rational function, plus a sum of integrals of the forms $I_r=\int y^{-1}x^r dx$, $J_r=\int (x-h)^{-r}y^{-1}dx$, and $L_r=\int (mx+n)(x^2+2px+q)^{-r}y^{-1}dx$ with $p^2-q<0$. The forms I_r and J_r are standard, but the author studies them in detail for the sake of completeness, and adds some complements to the standard theory. He then shows how one can express, in the real domain, L_r in terms of elementary functions and integrals of the forms I_0 and J_1. He shows by examples that his formulas may lead to simpler transformations and simpler final formulas than Legendre's method which is usually quoted. Finally he reduces the calculation of the elementary integral $\int R(x, (ax^2+bx+c)^{\frac{1}{2}})dx$ to the evaluation of simple standard forms, again using only real transformations.

R. P. Boas, Jr. (Evanston, Ill.).

Roelcke, W. Analytische Fortsetzung der Eisensteinreihen zu den parabolischen Spitzen von Grenzkreisgruppen erster Art. Math. Ann. **132** (1956), 121–129.

The author considers the Eisenstein-type series

$$E(\tau, s)=\sum y^{s/2}|m_1\tau+m_2|^{-s}\quad(\tau=x+iy,\ y>0)$$

summed over all parabolic cusps $-m_2/m_1$ in a given equivalence class for a fuchsian group of first kind. [See Petersson, Acta Math. **80** (1948), 23–63; MR **10**, 111.] He shows it can be extended analytically in Re $s>1$ with the exception of poles at $s=2$ and other s from 1 to 2. The method is to use the fact that E is an eigenfunction of $y^2\nabla^2 u+\lambda u=0$, with $\lambda=s(2-s)/4$. To use the Hilbert space $\|f\|=\int |f|^2 dxdy/y^2$ (over the fundamental domain), and to express u in terms of a resolvent, the author truncates E near the cusp points so as to remove terms of the Fourier series with the infinite behavior. [See the author's dissertation. S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. 1953/1955, 159–267 (1956). MR **18**, 476.] The author refers to a less general proof using the Epstein zeta-function, and a more general proof of A. Selberg by an undisclosed method, each producing an extension to the whole s-plane. *H. Cohn* (St. Louis, Mo.).

Rankin, R. A. The construction of automorphic forms from the derivatives of a given form. J. Indian Math. Soc. (N.S.) **20** (1956), 103–116.

Let $f(z)$ be a meromorphic automorphic form of arbitrary dimension k, belonging to the horocyclic group Γ, with the multipliers v. The author determines all polynomials $P(f, f_1, \cdots, f_n)$ in $f(z)$ and its first n derivatives (denoted by subscripts), which are themselves automorphic forms, $P\in\{\Gamma, k', v'\}$. Using relatively simple tools, by skillful but straightforward computations the following results are obtained: A term $Af_0^{\alpha_0}f_1^{\alpha_1}\cdots f_n^{\alpha_n}$ is said to be of degree r and weight s when $r=\sum_{j=1}^{n}\alpha_j$, $s=\sum_{j=1}^{n}j\alpha_j$. If P is automorphic then all its terms are of the same degree and weight. Define $h_r(z)=f_r(z)/\Gamma(k+r)r!$ and, for any integer $m\geqq 2$, set $\psi_m(z)=(-1)^{m-1}\|a_{ij}\|$, the terms of the determinant $\|a_{ij}\|$ being $a_{1j}=jh_j$ and, for $i\geqq 2$, $a_{ij}=h_{i-j+1}$ if $i-j+1\geqq 0$, $a_{ij}=0$ otherwise. In case k is a non-positive integer, set $k=1-N$, $g_r=f_r\Gamma(N-r)/r!$ and let $\phi_m(z)=(-1)^{m-1}\|b_{ij}\|$, the terms b_{ij} being defined as the a_{ij}'s, except that the g_r's now stand for the h_r's. With $\delta_2=g_1h_N+g_0h_{N+1}$ the main result of the paper can be stated as follows: If k is not a non-positive integer, then $f(z)$ and $\psi_m(z)$ $(2\leqq m\leqq n)$ form a basis for all automorphic forms that are rational functions of f and its first n derivatives. If k is a non-positive integer, then f, $f^{(N)}$ and χ_m (this is ψ_m formed from h_N instead of h_0) are automorphic for $N\geqq 1$, $m\geqq 2$, while δ_2 is automorphic for $N\geqq 2$ and ϕ_m for $N\geqq 3$, $2\leqq m<N$; these functions are polynomials in f and its first n derivatives and form a basis for all other such polynomials that are automorphic forms. Some examples are considered, where Γ is the full modular group. *E. Grosswald* (Philadelphia, Pa.).

Myrberg, P. J. Über automorphe Thetafunktionen bei Fuchsschen Gruppen vom Geschlecht null. Ann. Acad. Sci. Fenn. Ser. A. I. no. **223** (1956), 11 pp.

Let Γ be a fuchsian group with the principal circle $|z|=1$. The author defines automorphic theta functions to be functions $g(z)$ which are regular in $|z|<1$ and satisfy equations $g(Sz)=e^{u_s(z)}g(x)$ for all linear transformations $S\in\Gamma$. The exponents $u_S(z)$ are supposed to be integral functions which can be represented as linear combinations

$$u_S(z)=\sum_{\nu=1}^{N}a_\nu^{(S)}u_\nu(z)+a_{N+1}^{(S)}$$

of a finite number of them. Since the case $N=1$ was studied earlier [same Ann. no. **111** (1952); MR **14**, 262] the author now supposes $N>1$. For the sake of simplicity he takes $N=2$. Under some special assumptions (the genus of Γ is supposed to be zero and Γ is supposed to be generated by a finite number of elliptic linear transformations) he proves that any automorphic function belonging to the group Γ can be represented as a quotient of two automorphic theta functions, each of which is a product of primitive automorphic theta functions (an automorphic theta function is termed primitive if it has exactly one simple zero in the fundamental domain of Γ).

H. D. Kloosterman (Leiden).

Myrberg, P. J. Sur les fonctions automorphes à multiplicateurs exponentiels. J. Math. Pures Appl. (9) **35** (1956), 261–275.

Let G be a fuchsian group with the unit circle as its principal circle. Suppose that G contains no parabolic linear transformations and is generated by a finite number of elliptic and hyperbolic linear transformations. The author considers (meromorphic) automorphic theta functions having the property that

$$\text{(A)}\qquad g(Sz)=e^{\lambda_s u(z)+\eta_s}g(z)$$

for all $S\in G$, where $u(z)$ is regular in $|z|<1$. Then

$$\text{(B)}\qquad u(Sz)=\alpha_S u(z)+\beta_S.$$

He determines all functions $u(z)$ and $g(z)$ satisfying (B) and (A) respectively for all $S\in G$. Any automorphic function belonging to G can, except for an exponential factor, be represented as a quotient of two products of

primitive automorphic theta functions (having exactly one simple zero in the fundamental domain of G).
H. D. Kloosterman (Leiden).

Sandham, H. F. A square and a product of hypergeometric functions. Quart. J. Math. Oxford Ser. (2) 7 (1956), 153–154.

The author gives elementary proofs of Orr's theorem and of Clausen's theorem. He shows that Orr's theorem is a special case of Dougall's summation theorem for the ${}_7F_6$ hypergeometric series, and he then deduces Clausen's theorem by a simple argument based on hypergeometric functions of third order.
L. J. Slater.

Saran, Santi. Transformations of hypergeometric functions of three variables. Bull. Calcutta Math. Soc. 48 (1956), 9–23.

The author defines five hypergeometric functions of three variables and he gives double or triple integrals representing them. From these integrals he finds expansions of the triple hypergeometric functions in terms of Gauss functions thus

$$F_G(a_1', b_1, b_2, b_3; c_1, c_2'; x, y, z) = \sum_{m=0}^{\infty} \frac{(a_1, m)(b_2+b_3, m)}{(1, m)(c_2, m)}$$
$$\times {}_2F_1(a_1+m, b_1; c_1; x) {}_2F_1(-m, b_3; b_2+b_3; 1-z/y) y^m$$

and he deduces some new transformations of the functions, for example

$$F_M(a_1, a_2', b_1', b_2; c_1, c_2'; x, y, z) =$$
$$(1-x)^{-b_1} F_M(c_1-a_1, a_1', b_1, b_2; c_1, c_2; -x/(1-x), y, z/(1-x)).$$

He also proves as special cases several results concerning hypergeometric functions of two variables. Some of these have already been given by Bailey [Generalized hypergeometric series, Cambridge, 1935] or Appell and Kampé de Fériet [Fonctions hypergéométriques et hypersphériques, Gauthier-Villars, Paris, 1926], but others are new. There are several misprints in this paper. *L. J. Slater.*

See also: Epstein and French, p. 602; Mushiake, p. 622.

Sequences, Series, Summability

Vučković, Vladeta. Mercersche Sätze für nichtlineäre Mittel. Acad. Serbe Sci. Publ. Inst. Math. 10 (1956), 79–84.

Let $x_1, x_2, \cdots$ be a sequence of positive numbers, and let $X_1, X_2, \cdots$ be a sequence (determined by any process whatever) for which

$$\min_{1\le k\le n} x_k \le X_n \le \max_{1\le k\le n} x_k,$$
$$\liminf x_k \le \liminf X_n \le \limsup X_n \le \limsup x_n.$$

Supposing that $q\neq 1$, let

$$y_n = (x_n + qX_n)/(1+q) \quad (n=1, 2, \cdots).$$

Then the hypothesis $y_n \to y$ implies the Mercerian conclusion $x_n \to y$ provided $q > -1$ and the sequence $X_1, X_2, \cdots$ is either (1) monotone or (2) such that for each $n>1$ the differences $X_n - X_{n-1}$ and $y_n - x_n$ are both positive or both 0 or both negative. The last part of the hypothesis can be replaced by the hypothesis that, when n is sufficiently great, X_n lies in the closed interval with endpoints at X_{n-1} and y_n; and this together with convergence of y_n is enough to imply convergence of X_n and hence of x_n. Unlike the classic Mercerian theorems, this theorem applies to the nonlinear geometric and harmonic means.
R. P. Agnew (Ithaca, N.Y.).

van Dantzig, D. Complément à un problème de M. Karamata. Nieuw Arch. Wisk. (3) 4 (1956), 109–111.

In a previous note [same Arch. (3) 3 (1955), 89–92; MR 17, 29] the author answered a question of Karamata by showing that a real sequence $\{x_n\}$ which obeyed the restrictions $|\Delta x_n| \le 1/n$, $|\sum_1^n k^{-1} x_k| \le 1$, for $n=1, 2, 3, \cdots$, does not have to have a null subsequence $\{x_{n_k}\}$ with n_{k+1}/n_k bounded. In the present note, he completes this by proving the following. Let $\{x_n\}$ be a real sequence such that $\lim |\Delta x_n| = 0$, $|\sum_1^n \lambda_k x_k| \le 1$, where $k\lambda_k \ge \delta > 0$. Then, for any increasing sequence $\{\rho_n\}$ ($\lim \rho_n = \infty$), there is a subsequence $\{x_{n_k}\}$ converging to zero and such that $n_{k+1}/n_k \le \rho_k$.
R. C. Buck (Madison, Wis.).

Gatti, Stefania. Su un limite a cui tendono alcune medie. Metron 18 (1956), no. 1–2, 107–112.

Let $x_1, x_2, \cdots, x_n, \cdots$ be a sequence of positive numbers with $\lim_{n\to\infty} x_n = \lambda$. With C a real number not equal to 0 or 1, let E be defined by $C^E = n^{-1} \sum_{i=1}^n C^{x_i}$. It is proven that $\lim_{n\to\infty} C^E = C^\lambda$. Next let [Gini, Metron 13 (1938), no. 2, 3–22]

$$B_{dq}^{cp} = \left[\frac{\binom{n}{d} \sum_{i=1}^{\binom{n}{c}} P_i^c(x_i^p)}{\binom{n}{c} \sum_{i=1}^{\binom{n}{d}} P_i^d(x_i^q)} \right]^{(cp-dq)^{-1}}$$

in which $\sum_{i=1}^{\binom{n}{c}} P_i^c(x_c^p)$ is the sum of the $\binom{n}{c}$ different products of pth powers of c the first n x's and $cp - dq \neq 0$. It is also shown that

$$\lim_{n\to\infty} B_{dq}^{cp} = \lambda.$$

C. C. Craig (Ann Arbor, Mich.).

Goldberg, Karl. The formal power series for $\log e^x e^y$**.** Duke Math. J. 23 (1956), 13–21.

We operate in the free associative ring with the generators x and y over the field of rational numbers. The paper investigates the formal power series development of $\log e^x e^y$. The general term of the series is either of the form

$$W_x = W_x(s_1, s_2, \cdots, s_m) = x^{s_1} y^{s_2} \cdots (x\vee y)^{s_m},$$

where $s_1 s_2 \cdots s_m \neq 0$ and $(x\vee y)^{s_m} = x^{s_m}$ if m is odd and y^{s_m} if m is even. W_y is defined similarly. Let $c_x = c_x(s_1, s_2, \cdots, s_m)$ be the coefficient of W_x and $c_y = c_y(s_1, \cdots, s_m)$ that of W_y. The essential results of the paper are the following theorems. Theorem 1:

$$c_x(s_1, s_2, \cdots, s_m) = (-1)^{n-1} c_y =$$
$$\int_0^1 t^{m'} (1-t)^{m''} G_{s_1}(t) \cdots G_{s_m}(t)\, dt,$$

where $n = \sum_{i=1}^m s_i$, $m' = [\frac{1}{2}m]$, $m'' = [\frac{1}{2}(m-1)]$ and the polynomials $G_s(t)$ are defined by the recursion formula

$$sG_s(t) = \frac{d}{dt} t(t-1) G_{s-1}(t) \quad (s=2, 3, \cdots)$$

and $G_1(t) = 1$. Theorem 2: The generating function for the c_x is given by

$$\sum_{i=1}^m \sum_{s_i=1}^{\infty} c_x(s_1, \cdots, s_m) z_1^{s_1} \cdots z_m^{s_m} =$$
$$\sum_{i=1}^m z_i e^{m' z_i} \prod_{j\neq i} (e^{z_j} - 1)(e^{z_i} - e^{z_j})^{-1}.$$

In particular one obtains for $c_x(s_1, s_2)$ Theorem 3:

$$c_x(s_1, s_2)=\frac{(-1)^{s_1}}{s_1!s_2!}\sum_{i=1}^{s_2}\binom{s_2}{i}B_{s_1+s_2-i},$$

where B_k is the kth Bernoulli number.

C. Loewner (Palo Alto, Calif.).

Faulhaber, Gerhard. Äquivalenzsätze für die Kreisverfahren der Limitierungstheorie. Math. Z. **66** (1956), 34–52.

Das durch die Matrix

$$a_{n\nu}=\left(\frac{a}{\pi n}\right)^{\frac{1}{2}}e^{-a(\nu-n)^2/n}\ (n>0,\ \nu\geqq 0;\ a>0)$$

definierte Matrixverfahren F_a ist für $a=\frac{1}{2}$ und für Folgen $s_n=o(\sqrt{n})$ nach Hardy-Littlewood [vgl. etwa G. H. Hardy, Divergent series, Oxford, 1949, th. 151, p. 216; MR **11**, 25] äquivalent mit dem Borel-Verfahren. Nach Hyslop [Proc. London Math. Soc. (2) **41** (1936), 243–256] bleibt dies sogar richtig für alle Folgen mit $s_n=O(n^K)$ für ein $K>0$. Der Verf. baut die Hyslopsche Beweismethode aus und beweist damit für alle Folgen mit $s_n=O(n^K)$ die Aquivalenz von F_a (für jeweils ein a) mit den folgenden Verfahren: 1. Euler-Knopp E_p, 2. Taylor-Verfahren T_α,

$$a_{n\nu}=(1-\alpha)^{n+1}\binom{\nu}{n}\alpha^{\nu-n},\quad 0<\alpha<1,\quad \nu\geqq n,$$

3. den von Meyer-König [Math. Z. **52** (1949), 257–304; MR **11**, 242] eingeführten Verfahren S_β,

$$a_{n\nu}=(1-\beta)^{n+1}\binom{n+\nu}{\nu}\beta^\nu,\ 0<\beta<1.$$

Der Beweis erfolgt durch eine sorgfältige Entwicklung der Matrixelemente $a_{n\nu}$ dieser Verfahren in der Umgebung der Stelle $\nu=\nu(n)$ für die $a_{n\nu}$ (bei festem n) maximal wird. Ähnliche Untersuchungen werden für das von Meyer-König [ibid. **56** (1952), 179–205; MR **14**, 265] eingeführte Taylorsche Verfahren für Funktionen

$$T_b's=\frac{b^{n+1}}{n!}\int_0^\infty t^n e^{-bt}s(t)dt$$

durchgeführt. Insbesondere ergibt sich die Äquivalenz von T_α $(0<\alpha<1)$ mit T_b' $(b=(1-\alpha)/\alpha>0)$ für Folgen mit $s_n=O(n^K)$, falls $s(t)=s_n$, $n\leqq t<n+1$ gesetzt wird.

A. Peyerimhoff (Giessen).

Petersen, Gordon M. Inclusion between limitation methods. Math. Z. **65** (1956), 494–496.

Für zwei Limitierungsverfahren A und B bedeute $B\supset A$, dass jede beschränkte A-limitierbare Folge auch B-limitierbar ist, während eine beschränkte B-limitierbare, aber nicht A-limitierbare Folge existiert (strikte Inklusion). Verf. gibt einen neuen Beweis für einen Satz von A. Brudno [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **43** (1944), 183–185; Mat. Sb. N.S. **16(58)** (1945), 191–247; MR **6**, 150; **7**, 12]: Sind A und B permanente Matrixverfahren mit $B\supset A$, so gibt es ein permanentes Verfahren C mit $B\supset C\supset A$. Der Beweis wird erbracht durch Konstruktion einer Teilmatrix A' von A mit $A'\supset A$ (diese Bedingung fehlt im Lemma auf S. 494 — andernfalls wäre $A'=A$ trivialerweise möglich), die aber nicht alle B-limitierbaren Folgen limitiert. Die Matrix C wird aus den Zeilen von A' und B zusammengesetzt. (Im Beweis oben auf S. 495 muss $\lim\inf t_m{}^1=1$ durch $\lim\inf t_m{}^1=v<u$ ersetzt werden.)

A. Peyerimhoff (Giessen).

Ramanujan, M. S. Theorems on the product of quasi-Hausdorff and Abel transforms. Math. Z. **65** (1956), 442–447.

Die Arbeit liefert einen neuen Beitrag zu dem von Szász [Proc. Amer. Math. Soc. **3** (1952), 257–263; Ann. Soc. Polon. Math. **25** (1952), 75–84; MR **13**, 835; **15**, 26], Pati [Proc. Nat. Inst. Sci. India **20** (1954), 348–351; MR **16**, 124] und Rajagopal [J. Indian Math. Soc. (N.S.) **18** (1954), 89–105; MR **16**, 691] behandelten Fragekreis, wann für permanente Summierungsverfahren A und B die Beziehung $AB\supset A$ gilt. Die Gültigkeit dieser Beziehung wird nachgewiesen für $A=$Abelverfahren, $B=$quasi Hausdorffverfahren (in der Reihen-Reihen Form erklärt durch die Matrix $\binom{\nu}{n}\Delta^{\nu-n}\mu_n$, $\nu=n, n+1, \cdots$, wo $\{\mu_n\}$ eine reguläre Momentfolge ist), falls nur Reihen mit beschränkten Teilsummen zur Summation zugelassen werden.

A. Peyerimhoff (Giessen).

Rajagopal, C. T. A note on the oscillation of Riesz, Euler, and Ingham means. Quart. J. Math. Oxford Ser. (2) **7** (1956), 64–75.

Neuer Beweis für den folgenden, von W. B. Pennington [J. London Math. Soc. **27** (1952), 199–206; MR **13**, 738] und dem Verf. [Amer. J. Math. **69** (1947), 371–378, 851–852; **76** (1954), 252–258; MR **9**, 26, 278; **15**, 522] bewiesenen Satz: Existiert

$$\sigma_r(x)=\frac{A_r(x)}{x^r}=\frac{r}{x^r}\int_0^x(x-u)^{r-1}A(u)\,du$$

für alle $r>0$ und $x>0$ $(\sigma_0(x)=A(x)$, reell) und ist

$$\lim_{x\to\infty}{\sup\atop\inf}\sigma_r(x)={\bar\sigma_r\atop\underline{\sigma}_r},\ \lim_{r\to\infty}{\bar\sigma_r\atop\underline{\sigma}_r}=s\ \text{(endlich)},$$

so ist sogar $\bar\sigma_r=\underline{\sigma}_r=s$ für alle genügend grossen r. [Der entsprechende Satz für Cesàroverfahren stammt von J. E. Littlewood, J. London Math. Soc. **10** (1935), 309–310.] Zum Beweis wird gezeigt (Theorem 1_c), dass für genügend grosse r

$$\phi_r(y)=\frac{y^{r+1}}{\Gamma(r+1)}\int_0^\infty e^{-uy}A_r(u)du$$

für $y>0$ absolut konvergiert und $\lim_{y\to+0}\phi_r(y)=s$ ist. Daraus wird durch einen Taubersatz die Behauptung abgeleitet. In einem weiteren Beweis (Theoreme 1 und 2) wird mit der Karamataschen Methode direkt die Konvergenz von $\sigma_r(x)$ $(x\to\infty)$ bewiesen. Ein analoges Ergebnis wird für Eulerverfahren hergeleitet; dem Laplaceintegral entspricht dabei das Borelverfahren [das Ergebnis stammt im wesentlichen von W. Meyer König und K. Zeller, Math. Z. **59** (1953), 200–205; MR **15**, 305]. Ein weiteres Ergebnis über Inghamsche Mittel wird mit der Karamataschen Methode abgeleitet. *A. Peyerimhoff.*

Bojanić, Ranko. Quelques problèmes de sommation. Bull. Soc. Math. Phys. Macédoine **6** (1955), 9–17. (Serbo-Croatian. French summary)

Soit $y_n=cx_n+(1-c)x_{n-1}$, $c\neq 0$, et $c\neq 1$. On sait que la convergence de la suite $\{y_n\}$ entraîne celle de $\{x_n\}$ si $\mathrm{Re}\{c\}>\frac{1}{2}$. V. Vučković [Acad. Serbe Sci. Publ. Inst. Math. **8** (1955), 53–58; MR **17**, 961] a montré que ce résultat reste valable si l'on remplace la convergence par la sommabilité (N, p_n). L'auteur donne d'abord une simple démonstration de ce fait et ensuite, pour le procédé d'Euler il démontre que $(E, q)\lim y_n=\alpha$ entraîne $(E, q)\lim x_n=\alpha$, si $\mathrm{Re}\{c\}>1/(2|q+1|)$, q réel, $\neq -1$, tandis que pour le procédé de Borel le résultat correspondant subsiste si $\mathrm{Re}\{c\}>0$.

M. Tomić (Beograd).

See also: Thebault, p. 561; Butzer, p. 585; Boželov, p. 610; Ford, p. 626.

Approximations, Orthogonal Functions

Mauersberger, Peter. Die "Neumannsche Methode" zur Approximation einer durch Beobachtungen gegebenen Funktion und ihr Zusammenhang mit der mechanischen Quadratur nach Gauss-Jacobi. Z. Angew. Math. Mech. **36** (1956), 372–376. (English, French and Russian summaries)

Orthogonal polynomials applied to interpolation in n variables. *P. Davis* (Washington, D.C.).

Židkov, G. V. Remark on Bernstein polynomials. Grodnenskiĭ Gos. Ped. Inst. Uč. Zap. **1** (1955), 31–33. (Russian)

Denoting by $B_n(x)$ the nth degree Bernstein polynomial approximating $f(x)$, assumed continuously differentiable in $[0, 1]$ it is known that $B_n'(x) \to f'(x)$ as $n \to \infty$ [see, e.g., G. G. Lorentz, Bernstein polynomials, Univ. of Toronto Press, 1953; MR **15**, 217]. The present author sharpens this to $|B_n'(x)-f'(x)|<C(3/4n)^{\alpha/2}$, postulating that $f'(x)$ satisfies $|f'(x_1)-f'(x_2)| \leq C|x_1-x_2|^\alpha$, where $0<\alpha\leq 1$, $C>0$.
F. V. Atkinson (Canberra).

See also: Walsh, p. 569; Al-Salam, p. 570; Drazin, p. 570; Rudin, p. 587.

Trigonometric Series and Integrals

Aljančić, S.; Bojanić, R.; et Tomić, M. Sur le comportement asymptotique au voisinage de zéro des séries trigonométriques de sinus à coefficients monotones. Acad. Serbe Sci. Publ. Inst. Math. **10** (1956), 101–120.

The authors generalize known results connecting the asymptotic behavior of the coefficients of a sine series with the behavior of its sum in a neighborhood of 0. Let $L(x)$ be a positive continuous slowly increasing function ($L(tx)/L(x) \to 1$ as $x \to \infty$). Let $g(x)=\sum \lambda_n \sin nx$ with $\lambda_n \downarrow 0$. (1) If $0<\alpha<2$, $A>0$, then

$$g(x) \sim \{\tfrac{1}{2}A\pi/\Gamma(\alpha)\} \csc \tfrac{1}{2}\alpha\pi\, x^{\alpha-1}L(1/x)$$

implies $\lambda_n \sim An^{-\alpha}L(n)$; the converse is true, and holds for $1<\alpha<2$ if it is merely assumed that $\lambda_n>0$. (2) If $A>0$ and $\{\lambda_n\}$ is convex, then $\lambda_n \sim AL(n)$ implies $g(x) \sim Ax^{-1}L(1/x)$; the converse is true without the extra hypothesis on λ_n. *R. P. Boas, Jr.* (Evanston, Ill.).

Tomić, M. Sur la sommation de la série de Fourier d'une fonction continue avec le module de continuité donné. Acad. Serbe Sci. Publ. Inst. Math. **10** (1956), 19–36.

The author gives some conditions under which a triangular matrix sums the Fourier series of every function with a given modulus of continuity ω, generalizing results of an earlier paper [same Publ. **8** (1955), 23–32; MR **17**, 963]. Applications:

$$(1) \qquad s_n(x+\alpha_n)+s_n(x-\alpha_n) \to 2f(x)$$

uniformly, if and only if

$$\alpha_n=\frac{\pi k(n)}{2n+1}+\frac{1}{n \log n}\, o\{1/\omega(1/n)\}$$

with $h(n)$ odd and bounded. (2) If $\omega(\delta)=O(1/\log(1/\delta))$ the Fourier series of f converges uniformly, provided the Fourier coefficients c_n satisfy $|c_n| \leq F(n)/n$ with $F'(x)>0$, $F(n)=o(\log n)$, $F(n) \sim F(n+1)$. The author also corrects the proof of a theorem of the paper cited above on quasi convex uniform convergence factors [cf. the following review]. *R. P. Boas, Jr.* (Evanston, Ill.).

Bojanić, R. On uniform convergence of Fourier series. Acad. Serbe Sci. Publ. Inst. Math. **10** (1956), 153-158.

Let $\Lambda(t)$ be the partial sums of $\frac{1}{2}\lambda_0+\sum_1^\infty \lambda_\nu \cos \nu t$; let $\Omega(t)$ be continuous, decrease monotonically to 0 as $t \downarrow 0$, and satisfy $\Omega(0)=0$, $\Omega(x+y) \leq \Omega(x)+\Omega(y)$. Then the conditions

$$\int_0^{2\pi} \Big|\sum_{\nu=0}^{n} \Lambda_\nu(t)\Big| dt=O(n) \text{ and } \Omega(1/n)\int_0^{2\pi}|\Lambda_n(t)|dt=o(1)$$

are necessary and sufficient for the sequence $\{\lambda_n\}$, acting as a sequence of multipliers, to carry the Fourier series of any function whose modulus of continuity is Ω into a uniformly convergent series. This generalizes one of the theorems of Tomić in the paper reviewed above.
R. P. Boas, Jr. (Evanston, Ill.).

van der Corput, J. G. On the transformation of certain trigonometric sums. J. Analyse Math. **4** (1955/56), 236–245.

The author considers sums of the forms

$$\sum_{n \text{ in } (a,b)}^{(\lambda)} \exp(2\pi i\rho n)(n-\beta)^Q \sin^{-m}(\pi(\eta n+c)),$$

$$\sum_{n \text{ in } (a,b)}^{(\lambda)} \exp(2\pi i\rho n)(n-\beta)^Q \cos \pi(\eta n+c) \sin^{-m}(\pi(\eta n+c)),$$

where a, b integers, c, η, ρ, β real numbers, m small positive integer, $Q \geq 0$ small integer, $\lambda=0$ or 1 for $\lambda=0$ and $a<b$, $n=a+1/2, a+3/2, \cdots, b-1/2$, $\lambda=1$ and $a<b$, $n=a, a+1, \cdots, b$. The first and the last term in the sum has to be taken with the factor 1/2. It is shown that if either $|\rho/\eta|$ is a small positive integer or ρ/η, $a\eta+c$ and $b\eta+c$ are integers these sums can be represented non trivially as a linear combination of elementary sums. A sum is elementary if either $|b-a|$ small or ρ an integer because then the sums can be easily evaluated explicitly or asymptotically by means of the Euler Maclaurin sum formula. One sees immediately that these sums can be represented by a linear combination of sums of the form

$$\sum_{n \text{ in } (a,b)}^{(\lambda)} \exp(2\pi i\rho n)(n-\beta)^q f_\mu^{(q)}(\pi(\eta n+c)) \quad (\mu=0 \text{ or } 1),$$

where $f_0^{(q)}(x)=d^q(\operatorname{cosec} x)/dx^q$; $f_1^{(q)}(x)=d^q(\cot x)/dx^q$. Then the result follows from the identity

$$\begin{aligned}
&(2i)^{-q} \sum_{n \text{ in } (a,b)}^{(\lambda)} \exp(2\pi i(B\eta n+Cn+Bc)) \\
&\times (n-\beta)^Q f_\mu^{(q)}(\pi(\eta n+c)=\sum_{k=0}^{q}\binom{q}{k}(2i)^{-k}(A-B)^{q-k} \\
&\qquad \times \sum_{n \text{ in } (a,b)}^{(\lambda)} \exp(2\pi i[A\eta n+(n+Ac])(n-\beta)^Q f_\mu^{(k)}(\pi(\eta n+c)) \\
&\qquad\quad +\sum_{k=0}^{Q}\binom{Q}{k}(2i)^{-k}(b-\beta)^{Q-k} \\
&\times \sum_{n \text{ in } (A,B)}^{(\mu)} \exp(2\pi i(b\eta n+cn+bC))(n-B)^q f_\lambda^{(k)}(\pi(\eta n+c)) \\
&-\sum_{k=0}^{Q}\binom{Q}{k}(2i)^{-k}(a-\beta)^{Q-k} \sum_{n \text{ in } (A,B)}^{(\mu)} \exp(2\pi i(a\eta n+cn+aC)) \\
&\qquad \times (n-B)^q f_\lambda^{(k)}(\pi(\eta n+C))
\end{aligned}$$

where a, b, c, A, B, C, η, β, are arbitrary real numbers such that $b-a$, $B-A$ are integers q, Q arbitrary integers ≥ 0. If $\eta n+c$=integer, we replace the corresponding

terms in this relation (and correspondingly in the original sums) by a finite expression L where

$$L=\lim_{t\to c}[\exp(2\pi iBt)f_\mu^{(a)}(\pi(\eta n+t))-P(1/(t-c))],$$

where $P(x)$ is the polynomial given by the principal part of the Laurent expansion of the first term at c.

U. W. Hochstrasser (Washington, D.C.).

Hirokawa, Hiroshi. On the Cesàro summability of Fourier series. Kōdai Math. Sem. Rep. 7 (1955), 79–82.

Von C. Loo [Trans. Amer. Math. Soc. 56 (1944), 508–518; MR 6, 126] wurde bewiesen: Für Fourierreihen $\varphi(t)\sim\sum_1^\infty a_n\cos nt$ $(a_0=0)$ folgt für $\alpha>0$ aus

$$\sum_{\nu\le n}\binom{n-\nu+\alpha}{n-\nu}a_\nu=o(n^\alpha/\log n)(n\to\infty)$$

die Beziehung

$$\phi_{\alpha+1}(t)=\frac{1}{\Gamma(\alpha+1)}\int_0^t(t-u)^\alpha\varphi(u)du=o(t^{\alpha+1})(t\to 0).$$

Für diesen Sachverhalt wird ein neuer Beweis angegeben; ausserdem wird gezeigt, dass auch $1-<\alpha<0$ zulässig ist, falls noch $\sum_{\nu=n}^\infty|a_\nu|/\nu=O(n^{-\delta})$ gilt für $0<\delta<1$, $\alpha+\delta>0$. Zum Beweis für $\alpha>0$ wird $\phi_{\alpha+1}{}^*(t)=\int_0^t(t^2-u^2)^\alpha\varphi(u)du$ direkt abgeschützt — dabei wird eine Abschätzung von G. Sunouchi [Tôhoku Math. J. (2) 5 (1953), 198–210; MR 15, 788] über Integrale der Form

$$\int_0^t u^\beta(t^2-u^2)^{\alpha-1}\cos nu\,du$$

verwendet. Das Ergebnis folgt schliesslich daraus, dass nach K. Chandrasekharan und O. Szàsz [Amer. J. Math. 70 (1948), 709–729; MR 10, 369] $\phi_{\alpha+1}{}^*(t)=o(t^{2\alpha+1})$ die Beziehung $\phi_{\alpha+1}(t)=o(t^{\alpha+1})$ nach sich zieht. {Die Arbeit enthält einige Druckfehler; auf S. 80 muss in der Abschätzung von I_1'' die $\sum_{\nu=2}^N$ durch $\sum_M^{\nu-N+1}$ ersetzt werden.}

A. Peyerimhoff (Giessen).

Artémiadis, Nicolas K. Quelques théorèmes sur les transformées de Fourier et sur les coefficients des fonctions typiquement réelles. C. R. Acad. Sci. Paris 244 (1957), 544–547.

The author restates and generalizes an inequality for Fourier transforms that he gave in an earlier note [same C. R. 240 (1955), 1500–1502; MR 16, 817] and applies it to obtain some inequalities for the coefficients of typically-real power series in the unit disk. If the power series is $F(z)=z+\sum_2^\infty a_nz^n$, and $P(r)=\sum_0^k\gamma_nr^n$ is a polynomial with certain properties, he defines

$$\beta(x)=\tfrac12\sum_r^\infty a_pr^p$$

for $n-1<x\le n$, $\beta(x)=\beta(-x)$; and $\beta_1(x)=\gamma_0$ for $|x|\le 1$; γ_nr^n for $n<x\le n+1$ $(n=1,2,\cdots,k)$; 0 for larger x; β_1 even. He then applies one of his inequalities for Fourier transforms to β^*B_1, where B_1 is the Fourier transform of β_1. By specializing the polynomial he obtains inequalities like the following:

$$2+a_2-a_{\nu-1}-a_{\nu+1}+\tfrac12(a_{\nu-2}-a_{\nu+2})\ge 0;$$

if $F(r)\sim A/(1-r)$, then

$$|A+\tfrac12(a_{\nu-2}-a_{\nu+2})+a_{\nu+1}-a_{\nu-1}|\le A+2-a_2.$$

In a similar way he obtains an inequality for the coefficients of univalent star-shaped functions.

R. P. Boas, Jr. (Evanston, Ill.).

Integral Transforms

★ **Микусинский, Ян. [Mikusinskiĭ, Yan.] Операторное исчисление. [Operational calculus.]** Translation from the Polish by A. I. Plesner. Izdat. Inostr. Lit., Moscow, 1956. 366 pp. 15.25 rubles.

The original was reviewed in MR 16, 243.

Stein, Elias M. Interpolation of linear operators. Trans. Amer. Math. Soc. 83 (1956), 482–492.

Let T_z be an analytic family of linear transformations of simple functions on one measure space M to measurable functions on a second measure space N. Let us assume that for every pair of simple functions ϕ in M and ψ in N $\Phi(z)=\int_N(T_z\phi)\psi\,dn$ is analytic for $0<\mathrm{Re}\,z<1$ and continuous for $0\le\mathrm{Re}\,z\le1$, and that Φ does not grow too rapidly in this strip. If $\|T_{iy}f\|_{q_1}\le A_0(y)\|f\|_{p_1}$, $\|T_{1+iy}f\|_{q_2}\le A_1(y)\|f\|_{p_2}$ for every simple function f in M where $1\le p_1$, p_2, q_1, $q_2\le\infty$, then $\|T_tf\|_q\le A(t)\|f\|_p$, where $0<t<1$, $1/q=(1-t)/q_1+t/q_2$, $1/p=(1-t)/p_1+t/p_2$. [For a related result see Hirschman, J. Analyse Math. 2 (1953), 209–218; MR 15, 295, 1139.] Let E_n be n dimensional Euclidean space, and let $x=(x_1,\cdots,x_n)$, $\xi=(\xi_1,\cdots,\xi_n)$, $x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n$. Let $f(x)\in L^1(E_n)$,

$$F(\xi)=(2\pi)^{-n/2}\int_{E_n}e^{-ix\cdot\xi}f(x)dx,$$

and let

$$S_R(f)=\int_{|\xi|\le R}[1-(|\xi|/R)^2]^\delta e^{ix\cdot\xi}F(\xi)d\xi.$$

As an application of his interpolation theorem the author proves that $1<p<2$, $\delta>[(2/p)-1]\varkappa$, where $\varkappa=\frac12(n-1)$, then $\|S_Rf\|_p\le A\|f\|$, where A depends upon n and p and S but not upon R. The remainder of the paper is devoted to a generalization of Paley's inequalities for a uniformly bounded orthonormal set of functions.

I. I. Hirschman (St. Louis, Mo.).

See also: Il'in, p. 567; Tanimura, p. 581; Papoulis, p. 602.

Ordinary Differential Equations

★ **Feller, William. On generalized Sturm-Liouville operators.** Proceedings of the conference on differential equations (dedicated to A. Weinstein), pp. 251–270. University of Maryland Book Store, College Park, Md., 1956.

The author continues his study of the generalized Sturm-Liouville operators [Ann. of Math. (2) 61 (1955), 90–105; Comm. Pure Appl. Math. 8 (1955), 203–216; MR 16, 824, 927]. He again considers the class of non-trivial linear operators Ω whose domain and range are in the space of continuous functions and which are of local character and have the minimum property.

The work of the previous papers is greatly reduced by assuming Ω to be regular in the sense that there are two linearly independent solutions of $\Omega w=0$ in the neighborhood of each point. It is shown that there exist strictly increasing functions $s(x)$ and $m(x)$, s being continuous, such that if $w(x)$ is any non-vanishing (real or complex) solution of $\Omega w=0$, the operator Ω has the representation

$$\Omega f=\frac{1}{w}D_m\Big[w^2D_s\Big(\frac{f}{w}\Big)\Big].$$

Here D_m and D_s are generalized derivatives with respect to m and s respectively.

Using this representation, the Weyl limit point limit-circle classification and relayed existence theorems are developed in a Hilbert space whose norm is $\int f(x)^2 dm(x)$ so that Ω is self-adjoint in this space.

The following typographical errors were noted: (a) in (5.3) the inequality is reversed; (b) in (7.11) and (7.19) the lower limits of the first integrals should be s rather than s_0; (c) the inequality two lines below (9.3) should be $\Omega F(\xi) \geq 0$. *H. Weinberger* (College Park, Md.).

Utz, W. R. Boundedness and periodicity of solutions of the generalized Liénard equation. Ann. Mat. Pura Appl. (4) 42 (1956), 313–324.

The author considers equations of the form

$$\ddot{x}+\phi(x, \dot{x})+g(x)=e(t),$$

which have been studied extensively [for summaries, see, e.g., M. L. Cartwright, E. T. Copson and J. Greig, Advancement Sci. 6 (1949), no. 21; MR 11, 32, and G. Sansone, Atti 4⁰ Congresso Un. Mat. Ital., Taormina, 1951, v. 1, Edizioni Cremonese, Roma, 1953, pp. 186–217; MR 15, 32]. He proves: (i) If $e(t)=0$, and $G(x)=\int_0^x g(u)du$ exists for every finite x and $\to\infty$ with $|x|$, and if $\phi(x, \dot{x})\equiv f(\dot{x})$ where $uf(u)\geq 0$ for all u or $\phi(x, \dot{x})\equiv f(x)\dot{x}$ where $f(u)\geq 0$ for all u, in either case $f(u)$ being integrable over every finite interval, then all solutions are bounded as $t\to\infty$. (ii) If $e(t)\not\equiv 0$, and $g(x)=x$, and if $\phi(x, \dot{x})$ satisfies the latter hypotheses, the same conclusion holds provided there exists one solution that is bounded as $t\to\infty$. The author also obtains results on the behavior of solutions as $t\to\infty$ under conditions in general too weak to ensure boundedness and he extends a theorem of McHarg [J. London Math. Soc. 22 (1947), 83–85; MR 9, 435] on the existence of an infinity of periodic solutions when $e(t)\equiv 0$. All proofs depend on considerations of the integrated equation or of positive definite gauge functions.

H. A. Antosiewicz (Washington, D.C.).

Pipes, Louis A. Stability of periodic time-varying systems. Math. Mag. 30 (1956), 71–80.

The author describes two methods for discussing the behavior of the solutions of the equation $x''-F(t)x=0$ where $F(t)$ is a periodic function of t. One method involves a matrix multiplication technique described by the author in a previous paper [J. Appl. Phys. 24 (1953), 902–910; MR 15, 128] and the other is an approximate method originally given by Lord Rayleigh [Phil. Mag. (5) 24 (1887), 145–159]. These methods are illustrated for the problem of the "inverted pendulum." *J. K. Hale.*

Lehrer, Y. Note on simultaneous linear differential equations with constant coefficients. Proc. Cambridge Philos. Soc. 53 (1957), 257–258.

The author writes the solution of the differential system

$$U'+cU=V,\ U(0)=U_0$$

(where U, V are column vectors and c is a square constant matrix of special form) as

$$U=\sum_{k=0}^{m-1} c^k\Big\{f_k(-t)U_0+\int_0^t f_k(\theta)Vd\theta\Big\},$$

where the f_k form a fundamental system of solutions of the equation

$$z^{(m)}+p_{m-1}z^{(m-1)}+\cdots+p_0z=0,$$

subject to the initial conditions

$$[f_j^{(k)}(t)]_{t=0}=\delta_{jk}\ (j, k=0, \cdots, m-1),$$

and the polynomial

$$p(\lambda)=\lambda^m+p_{m-1}\lambda^{m-1}+\cdots+p_0$$

is the characteristic polynomial of c. *W. J. Coles.*

Hartman, Philip. Self-adjoint, non-oscillatory systems of ordinary, second order, linear differential equations. Duke Math. J. 24 (1957), 25–35.

A differential equation $x''+f(t)x=0$ is called non-oscillatory on $0\leq t<\infty$ if no (non-trivial) solution has arbitrarily large zeros. This notion is generalized to a self-adjoint system of equations $x''+F(t)x=0$ in which x is a vector and $F(t)$ is a continuous Hermitian matrix on $0\leq t<\infty$. Results concerning solutions of this system (and, more generally, of the system $(Px')'+Fx=0$ in which $P(t)$ is a continuous Hermitian positive definite matrix), analogous to known facts concerning solutions in the particular cases that $F(t)$ is a scalar or a real Hermitian non-positive definite matrix, are obtained.

C. R. Putnam (Lafayette, Ind.).

Bellman, Richard. Notes on matrix theory. X. A problem in control. Quart. Appl. Math. 14 (1957), 417–419.

Let A and B be $n\times n$ constant matrices, and let an n-dimensional vector ($n\times 1$ matrix) x be determined by the equations $dx/dt=Ax$, $x(0)=c$. In this note it is shown that if the characteristic roots of A have negative real parts, the integral

$$\int_0^\infty (x, Bx)dt$$

can be evaluated without having to solve the differential equation explicitly. The result is of value in the theory of control processes. *L. A. MacColl.*

Bellman, Richard. Notes on control processes. I. On the minimum of maximum deviation. Quart. Appl. Math. 14 (1957), 419–423.

Let x be a vector function of t defined by the equations $dx/dt=\varphi(x, y)$, $x(0)=c$, where y is a given vector function of t and c is a constant vector. Also let z be a fixed vector function of t. It is assumed that the vector space is normed; and it is required to determine y so as to minimize the quantity $\max_{0\leq t\leq T}\|x-z\|$. The author describes, rather too concisely, a procedure by which the required y can be found, by solving a non-linear partial differential equation or a recurrence relation. *L. A. MacColl.*

Schäffer, Juan Jorge. Analytische Parameterabhängigkeit der fastperiodischen Lösungen von nichtlinearen Differentialgleichungen. Rend. Circ. Mat. Palermo (2) 5 (1956), 204–236.

Let B be the space of m by n matrices whose elements are real-valued uniformly almost periodic (u.a.p.) functions of the real variable t. If $\|A\|=\sup_{\|x\|=1}\|Ax\|$ where $\|x\|$ is the Euclidean norm of the vector $x\in R^n$ (real, n-dimensional, linear space), then $|A(t)|=\sup_t \|A(t)\|$ is a norm on B. Consider the space $\sum_{m\times n}(r)$ of matrices of the form $A(t, \lambda)=\sum_0^\infty A_i(t)\lambda^i$ such that $\sum_0^\infty |A_i(t)|r^i<\infty$, where the $A_i\in B$. $A(t, \mu)$ is said to be a u.a.p. function of t uniformly analytically dependent (u.a.d.) on μ in an (open) interval I if for every $\mu_0\in I$ there is an $r=r(\mu_0)>0$ such that $B(t, \lambda)=A(t, \mu_0+\lambda)\in\sum_{m\times n}(r)$. The principal

result of this paper is concerned with the equation

(1) $$\frac{dx}{dt}+f(x, p(t, \lambda))=0,$$

where $f(x, y)$ is analytic in $(x, y) \in R^n \times R^m$ in some suitable region and $p(t, \lambda) \in \sum_{m\times 1}(r)$. Let $x_0(t)$ be a u.a.p. solution of (1) for $\lambda=0$ and suppose

(2) $$\frac{dz}{dt}+\frac{\partial}{\partial x}f(x_0(t), p(t, \lambda))z+q(t)=0$$

$(\partial f(x, y)/\partial x$ being the Jacobian matrix) possesses for $\lambda=0$ at least one bounded solution for $t\geq 0$ for every bounded continuous $q(t)$ or at least one bounded solution for $t\leq 0$ for every bounded continuous $q(t)$. Then there exists an $r_0>0$ and $x(t, \lambda) \in \sum_{n\times 1}(r_0)$ which is a solution of (1) for $|\lambda|\leq r_0$ such that $x(t, 0)=x_0(t)$. Also if $p(t, \lambda)$ is u.a.d. on λ in I, if (1) possesses exactly one u.a.p. solution $x(t, \lambda)$ for each $\lambda \in I$ and if the solutions of (2) satisfy the above boundedness conditions for each $\lambda \in I$, then $x(t, \lambda)$ is u.a.d. on λ in I. The author also obtains some results relating the Fourier exponents of $x(t, \lambda)$ and $p(t, \lambda)$.

C. E. Langenhop (Ames, Iowa).

Göransson, K.; and Hansson, L. An experimental investigation of subharmonic oscillations in a nonlinear system. Kungl. Tekn. Högsk. Handl. Stockholm no. 97 (1956), 16 pp.

The results of Lundquist [Quart. Appl. Math. 13 (1955), 305–310; MR 17, 369] are verified experimentally for small perturbations from the linear. Experimental results for the strong non-linear case are also given.

N. Levinson (Cambridge, Mass.).

Faure, Robert. Sur certaines solutions périodiques d'équations différentielles non linéaires. Cas des vibrations forcées. Influence de la fréquence. Ann. Mat. Pura Appl. (4) 42 (1956), 165–188.

Theorem. Consider (1) $y''+f(y)y'+ky=e(t)$, where $e(t)$ is periodic of period $T=2\pi/\omega$, $e(t)=\sum \alpha_n e^{in\omega t}$, $\alpha_0=0$, $\sum|\alpha_n|=\eta<\infty$, and $f(y)$ is an analytic function with $f(0)\neq 0$. For a given ω and for η sufficiently small, the equation (1) possesses a periodic solution of period T and the solution, together with its first and second derivatives, has absolutely convergent Fourier series. The solution and its first derivative approach zero as $\eta\to 0$. For any η, there always exists a periodic solution of period T provided that ω is sufficiently large and the solution together with its first derivative approach zero as $\omega\to\infty$. The author proves the above theorem by rewriting (1) as $y''+ay'+ky=e(t)+g(y)y'$, $g(y)=a-f(y)$, $a=f(0)\neq 0$. Defining functions u_n by the recurrence relation

$$u_n''+au_n'+ku_n=e(t)+g(u_{n-1})u_{n-1}',$$

he shows that the functions u_n converge to a periodic solution of (1) provided that a certain inequality involving η and ω is satisfied. A similar theorem is given for the equation $y''+f(y)y'+g(y)=e(t)$, but it is more difficult to form the recurrence relation for the functions u_n. The paper also contains some analogous theorems for differential difference equations.

J. K. Hale.

Reissig, Rolf. Über die Stabilität erzwungener Bewegungen. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe 5 (1955/56), 103–105. (Russian, English and French summaries)

When examining stationary movements one must also be interested in their stability, for only stable movements are realizable. Extreme stability is then evident when the course of movement considered will withstand all disturbances. Using the methods of E. Trefftz [Math. Ann. 95 (1925), 307–312], the author shows that periodic excitement of an extremely stable system always leads to the same forced vibration.

J. K. Hale.

Manaresi, Gabriella. Sopra alcune limitazioni per l'ampiezza delle oscillazioni non-lineari. Atti Accad. Sci. Ist. Bologna. Cl. Sci. Fis. Rend. (11) 2 (1954–55), 184–189.

The author considers the equation

(1) $$\ddot{x}+\dot{F}(x)+\omega_0^2x=0,$$

where $F(x)=\int_0^x f(x)dx$, $f(x)$ is symmetric and negative for $|x|<\delta$, $\delta>0$ and $f(x)=\alpha>0$ otherwise. Then there exists an h such that $F(h)=F(-h)=0$. Following the method of S. Mizohata and M. Yamaguti [Mem. Coll. Sci. Univ. Tokyo. Ser. A, Math. 27 (1952), 109–113; MR 14, 874] the author obtains bounds for the free oscillation of (1) which are improvements of some estimates obtained by E. Cartan [Ann. Postes Télégraphes Téléphone 14 (1925), 1196–1207].

J. K. Hale (St. Paul, Minn.).

Sansone, G.; e Conti, R. Determinazione dell'integrale positivo minimo nell'equazione di M. Hukuhara. Rev. Un. Mat. Argentina 17 (1955), 213–216 (1956).

Consider the equation (1) $xy'=Ay^k+B(x)$, where $A>0$, $k>1$ are constants, and $B(x)$ is a continuous function in $0\leq x\leq x_0$, $(x_0>0)$, $B(0)=0$; $B(x)>0$ for $0<x\leq x_0$. Assume that (1) has a solution $y(x)$ with the properties (2) $y(x)>0$ for $0<x<x_0$ and $\lim_{x\to 0+} y(x)=0$. M. Hukuhara [Proc. Phys.-Math. Soc. Japan (3) 21 (1939), 183–190] has given a necessary and sufficient condition that (1) have a solution with the properties (2). The authors show that the set of all solutions (2) has a minimum and give a method of successive approximations for calculating this minimum.

J. K. Hale (St. Paul, Minn.).

See also: Seidenberg, p. 568; Zubov, p. 578; Bellman, p. 582.

Partial Differential Equations

★ **Sneddon, Ian N. Elements of partial differential equations.** McGraw-Hill Book Company, Inc., New York-Toronto-London, 1957. ix+327 pp. $7.50.

In the author's own words: "The aim of this book is to present the elements of the theory of partial differential equations in a form suitable for the use of students and research workers whose main interest in the subject lies in finding solutions of particular equations rather than in the general theory". An attempt has been made to touch upon subjects from the entire field of partial differential equations. There are six chapters in all. Chapter 1 is introductory in character and deals with ordinary differential equations in more than two independent variables. Of special interest here is the treatment of Pfaffian differential forms, Carathéodory's theorem on integrability, and its application to thermodynamics. Partial differential equations of the first order occupy chapter 2. Among the subjects covered are Cauchy's problem for first order equations, linear and nonlinear equations, Cauchy's method of characteristics, Lagrange's and Charpit's methods, and applications, for example to stochastic processes and to birth and death processes in

bacteria. Chapter 3 is concerned with a preliminary discussion of second (and higher) order equations, preparatory to the last three chapters, which consider in more detail the three main types of second order equations. One finds a treatment of linear partial differential equations with constant coefficients, canonical forms for second order equations, characteristics of equations in two and three independent variables, Riemann's method of integration, and integral transforms. Chapter 4 deals with Laplace's equation, the prototype of the elliptic second order linear partial differential equations. The Dirichlet, Neumann and mixed boundary value problems are dealt with, Green's functions, Kelvin's inversion theorem, and allied topics. The typical hyperbolic equation, the wave equation, constitutes the subject matter of chapter 5. One, two, and three space dimensions are considered; the Riemann-Volterra solution of the one dimensional equation, and Helmholtz', Weber's and Kirchhoff's general solutions in two and three dimensions. There is a brief introduction to Marcel Riesz' method of solution, based on Riesz' generalization of the Riemann-Liouville integral. The concluding chapter 6 has the heat equation, the typical parabolic equation, as its topic. There is an appendix on systems of surfaces, meant to provide a brief outline of some of the properties of systems of surfaces used in chapter 2. The use of this book as a text is facilitated by the presence of a large number of problems, together with the solution of all odd numbered ones. *J. B. Diaz* (Cambridge, Mass.).

Prodi, Giovanni. Sul primo problema al contorno per equazioni a derivate parziali ellittiche o paraboliche, con secondo membro illimitato sulla frontiera. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. **90** (1956), 189–208.

Let A denote a domain in the xy-plane and FA its boundary. For a point P in A let the minimum distance between P and FA be indicated by $\delta(P)$. The coefficients of the elliptic operator

$$L(u)=au_{xx}+2bu_{xy}+cu_{yy}+du_x+eu_y+gu \qquad (ac-b^2>0,\ g\leqq 0)$$

are assumed continuous functions of (x, y) on $A\cup FA$. It is also assumed that on each closed subset of A these coefficients as well as another function $f=f(P)$ satisfy Hölder conditions. Under relatively light restrictions on the boundary of A the author proves that the Dirichlet problem

$$L(u)=f \text{ on } A,\quad u=0 \text{ on } FA$$

admits a unique solution $u=u(x, y)$ of class C^2 on A and C^0 on the closure of A provided there exist positive constants K and $\mu<2$ such that $|f(P)|\leqq K(\delta(P))^{-\mu}$.

Similar reults are obtained relative to the first boundary problem for the parabolic equation $L(u)-u_t=f(P, t)$. Here the coefficients $a, b, \cdots, g$ may depend on t $(0\leqq t\leqq T)$ as well as (x, y) and the function f must satisfy a restriction of the type $|f(P, t)|\leqq K\{(\delta(P))^{-\mu}+t^{-\frac{1}{2}\mu}\}$. {Related results on the parabolic equation

$$u_t=u_{xx}+F(x, t, u, u_x)$$

may be found in the author's papers in same Rend. (3) **17**(86) (1953), 3–26, 27–47; MR **16**, 259. Condition H_2 in the review of this paper is misprinted. It should read: "H_2) For each interval $-v\leqq u\leqq v$ there exists a constant K_v such that $|F(x, t, u, u_x)|\leqq K_v\{(\psi(x, t))^\gamma+|u_x|^\gamma\}$, where γ is a positive constant less than 2."}
F. G. Dressel (Durham, N.C.).

★ **Transactions of the Symposium on Partial Differential Equations held at the University of California, at Berkeley, June 20–July 1, 1955.** Sponsored by Office of Naval Research, Univ. of California, Berkeley, Calif., Univ. of Kansas, Lawrence, Kan., and Amer. Math. Soc.; Editorial Committee, N. Aronszajn — A. Douglis — C. B. Morrey, Jr. Interscience Publishers, Inc., New York, 1956. vi+334 pp. $6.50.

This book is a reprint of Comm. Pure Appl. Math. IX, 3 (1956), whose 28 articles are being currently reviewed in MR.

Zubov, V. I. Representation of solutions of systems of differential equations in the neighborhood of a singular point. Dokl. Akad. Nauk SSSR (N.S.) **109** (1956), 1095–1097. (Russian)

The author considers a system of partial differential equations of the form

$$(1)\quad \sum_{s=1}^{n}\frac{\partial z_j}{\partial x_s}\Big(\sum_{i=1}^{n} p_{si}(t)x_i+X_s\Big)+\frac{\partial z_j}{\partial t}= \sum_{i=1}^{k} q_{ji}(t)z_i+\sum_{i=1}^{n} r_{ji}(t)x_i+Z_j \quad (j=1, \cdots, k,)$$

where

$$X_s=\sum_{\sum_1^n m_i+\sum_1^k n_i\geqq 2} p_s^{(m_1,\cdots,m_n,n_1,\cdots,n_k)}(t)x_1^{m_1}\cdots x_n^{m_n}\cdot z_1^{n_1}\cdots z_k^{n_k} \qquad (s=1, \cdots, n),$$

$$Z_j=\sum_{\sum_1^n m_i+\sum_1^k n_i\geqq 2} Q_j^{(m_1,\cdots,m_n,n_1,\cdots,n_k)}(t)x_1^{m_1}\cdots x_n^{m_n}\cdot z_1^{n_1}\cdots z_k^{n_k} \qquad (j=1, \cdots, k).$$

The functions

$$p_{sl}(t),\ q_{ij}(t),\ r_{sl}(t),\ P_s^{(m_1,\cdots,m_n,n_1,\cdots,n_k)}(t),\ Q_j^{(m_1,\cdots,m_n,n_1,\cdots,n_k)}(t) \qquad (s, l=1, \cdots, n;\ i, j,=1, \cdots, k)$$

are assumed real, continuous, and bounded for $t\geqq 0$. The author states that if suitable relations hold between the $\{p_{sl}\}$ and the $\{q_{ij}\}$, there will exist holomorphic functions $z_j(x_1, \cdots, x_n, t, c_1, \cdots, c_\beta)$ $(j=1, \cdots, k)$ depending on β arbitrary constants, which are solutions of (1) in a region of the form $|x_s|\leqq x_0(t)\neq 0$, $t\geqq 0$, $|c_\sigma|\leqq c_0$. These functions appear as infinite sums of homogeneous forms in $x_1, \cdots, x_n$, with coefficients depending on t and on the $c_1, \cdots, c_\beta$.

This result is applied to a system of ordinary differential equations

$$(2)\quad z\frac{dy_s}{dz}=\sum_{i=1}^{n}\tilde{p}_{si}(z)y_i+\tilde{p}_s(z)z+Y_s(z, y_1, \cdots, y_n) \qquad (s=1, \cdots, n),$$

where

$$Y_s=\sum_{m+m_1+\cdots+m_n\geqq 2}\tilde{p}_s^{(m,m_1,\cdots,m_n)}z^m y_1^{m_1}\cdots y_n^{m_n},$$

and $\tilde{p}_{si}$, $\tilde{P}_s$, $\tilde{P}_s^{(m,m_1,\cdots,m_n)}$ $(s, i=1, \cdots, n)$ are real, continuous, and bounded for $0<z\leqq 1$. The author states that under suitable restrictions on the $\{\tilde{p}_{si}\}$, (2) admits a family of solutions of the form

$$y_s=\sum_{m+\sum_{i=1}^{\beta} m_i\geqq 1}\tilde{K}_s^{(m,m_1,\cdots,m_\beta)}(z)z^{m+\sum_{i=1}^{\beta} m_i\lambda_i}c_1^{m_1}c_2^{m_2}\cdots c_\beta^{m_\beta} \qquad (s=1, \cdots, n)$$

convergent for $|z|\leqq z_0$, $|c_\sigma|\leqq c_0$, $\sigma=1, \cdots, \beta$. *R. Finn.*

Foias, Ciprian; Gussi, George; et Poenaru, Valentin. Une méthode directe dans l'étude des équations aux dérivées partielles hyperboliques, quasilinéaires en deux variables. Math. Nachr. **15** (1956), 89–116.

The article deals with the initial value problem (on $t=0$) for the (semi-linear) hyperbolic system

$$\frac{\partial u_i}{\partial t}+\sum_{j=1}^{n} a_{ij}(t,x)\frac{\partial u_j}{\partial u}=f_i(t,x,u_1,\cdots,u_n)\ (i=1,2,\cdots,n)$$

it being assumed that the proper values of the matrix (a_{ij}) are real and distinct. The coefficients $a_{ij}(t,x)$, $f_i(t,x,u)$ are continuous and satisfy additional conditions, to be elaborated below, but need not be differentiable. Concerning each function $a_{ij}(t,x)=a(t,x)$, it is assumed that

$$\limsup |h^{-1}\{a(t,x+h)-a(t,x)\}|\leq k(t),$$
$$\limsup |h^{-1}\{a(t+h,x)-a(t,x)\}|\leq l(t)\ (h\to 0),$$

where $k(t)$, $l(t)$ are summable. Concerning the set of functions $f_i(t,x,u,\cdots,u_n)=f_i(t,x,u)$, it is assumed that $|f_i(t,x,u)-f_i(t,x,v)|\leq K_i(t)\omega(\sum |u_k-v_k|)$, where the $K_i(t)$ are summable and $\omega(z)$ is a positive non-decreasive function such that for any $\eta>0$, $\int_0^\eta dz/\omega(z)=+\infty$. In this context it is proved that if the initial value problem has a C^1 solution, then that solution is unique. When the matrix (a_{ij}) is diagonal, more specific results can be obtained. Let $a_{ij}(t,x)=\lambda_i(t,x)\delta_{ij}$, with $\lambda_i(t,x)$ continuous and such that $d\xi_i/dt=\lambda_i(t,\xi_i(t))$ has a unique solution $x=\xi_i(t,t_0,x_0)$ passing through each point (t_0,x_0). By integrating the ith equation along the corresponding characteristic curve $x=\xi_i(t,t_0,x_0)$, the hyperbolic differential system is replaced, in the familiar manner, by a system of integral equations. A continuous solution $u_1(t,x),\cdots,u_n(t,x)$ of the integral system is termed a generalized solution of the differential system. Such a solution is shown to be a generalized solution also in the sense of Soboleff. Under the conditions described concerning the functions $f_i(t,x,u)$, it is proved that the Cauchy problem has (locally) a unique generalized solution, and that the solution varies continuously with the initial data. There are also some theorems regarding continuation of solutions and behavior in the large.

R. W. McKelvey (Boulder, Colo.).

Hersch, Joseph. Une équation aux différences pour le calcul approché des fréquences propres d'une membrane (méthode récurrente). C. R. Acad. Sci. Paris **243** (1956), 1475–1478.

The approximation of the eigenvalues of a membrane is considered, where the differential equation

$$\Delta u+\lambda u=0$$

is approximated by the finite difference eigenvalue equation

$$L_h[u]\equiv A_h u(x_0,y_0)-u(x_0+h,y_0)-u(x_0-h,y_0)$$
$$-u\Big(x_0+\frac{1}{2}h,\ y_0+\frac{\sqrt{3}}{2}h\Big)-u\Big(x_0+\frac{1}{2}h,\ y_0-\frac{\sqrt{3}}{2}h\Big)$$
$$-u\Big(x_0-\frac{1}{2}h,\ y_0+\frac{\sqrt{3}}{2}h\Big)-u\Big(x_0-\frac{1}{2}h,\ y_0-\frac{\sqrt{3}}{2}h\Big)=0$$

on a grid consisting of equilateral triangles of side h. The classical approximation λ_h to λ is obtained by setting

$$A_h=6-\tfrac{3}{2}\lambda_h h^2.$$

The author uses an argument concerning the change of the finite difference equation when the grid is refined to suggest approximating λ by $\tilde{\lambda}$ defined by

$$A_h=2+4\cos(\tfrac{1}{2}h\sqrt{(3\tilde{\lambda})}).$$

A different argument suggests approximating λ by $\mathring{\lambda}$ defined by

$$A_h=6J_0(h\sqrt{\mathring{\lambda}}).$$

These expressions differ only by terms of $O(h^6)$. It appears to the reviewer that the last two definitions give an improvement over the first one because $L_h[u]+\Delta u+\lambda u$ is $O(h^4)$ rather than $O(h^2)$. Thus, it seems that one could use any λ^* defined by

$$A_h=6-\tfrac{3}{2}\lambda^* h^2+\tfrac{3}{32}\lambda^{*2}h^4+O(h^6).$$

The author shows by several numerical examples that his method substantially improves the approximation of the lowest eigenvalue. He remarks that the method probably does not give good results for higher eigenvalues.

H. F. Weinberger (College Park, Md.).

Landkof, N. S. Some new properties of the set of irregular points for the generalized Dirichlet problem. Har'kov. Gos. Univ. Uč. Zap. **29**=Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) **21** (1949), 145–163. (Russian)

Let Ω be a bounded region in 3-dimensional space and let F be the boundary of Ω. The author gives examples to show that the set of irregular boundary points of F relative to Ω may comprise a linear continuum and that the points of such a continuum may be accessible from outside Ω. A region Ω is said to be of type (Γ) if every open Jordan arc on F is accessible from outside by a Jordan surface. The main result of the paper is the theorem that if Ω is a region of type (Γ) and L a plane Jordan curve lying on F, then the irregular points of L relative to Ω form a set of second category.

M. G. Arsove.

Stampacchia, Guido. Su un problema relativo alle equazioni di tipo ellittico del secondo ordine. Ricerche Mat. **5** (1956), 3–24.

Sia D un dominio limitato dello spazio $S_{(n)}$ e sia

$$\mathscr{F}D=\mathscr{F}_1D+\mathscr{F}_2D,$$

dove $\mathscr{F}_1D$ ed $\mathscr{F}_2D$ sono due varietà $(n-1)$ dimensionali dotate di normale lipschitziana e rappresentabili, ognuna, mediante un numero finito di rappresentazioni parametriche locali $x_i=x_i(\xi_1,\xi_2,\cdots,\xi_{n-1})$. Sia inoltre $\mathscr{W}$ una varietà regolare $(n-1)$-dimensionale che divida D in due domini D_1 e D_2, e sia $\mathscr{W}\cdot D=\Sigma$.

Si considera il problema consistente nel determinare due funzioni $u_1(x)$, $u_2(x)$, definite rispettivamente in D_1 e D_2, e tali che $u_s(x)$ sia soluzione, in $D_s-\mathscr{F}D_s$, dell'equazione ellittica autoaggiunta.

$$(1)\qquad E_s(u_s)=\sum_{i=1}^{n}\frac{\partial}{\partial x}\sum_{j=1}^{n}a_{ij}^{(s)}\frac{\partial u_s}{\partial x_j}=c_su_s+f_s\quad (s=1,2)$$

e verifichi le condizioni al contorno:

$$u_s=\alpha,\ \text{per } x\in\mathscr{F}_1D\cdot\mathscr{F}D_s,$$
$$(2)\qquad \frac{du_s}{dv_s}=\varphi u_s+\beta,\ \text{per } x\in\mathscr{F}_2D\cdot\mathscr{F}D_s$$

essendo α una funzione definita su $\mathscr{F}_1D$, φ e β due funzioni definite su $\mathscr{F}_2D$, v_s la conormale orientata verso l'interno di D_s, ed avendo posto

$$\frac{du_s}{dv_s}=\sum_{i,j}a_{ij}^{(s)}\cos(n,x_j)\partial u_s/\partial x_i.$$

Alle funzioni u_1 ed u_2 si prescrivono poi le condizioni di raccordo:

$$(3)\qquad u_1=u_2, \frac{du_1}{dv_1}=-\frac{du_2}{dv_2}, \text{ per } x \in \Sigma.$$

Servendosi dei metodi del Calcolo delle Variazioni, ed in ipotesi molto generali sui coefficienti e sui dati al contorno, l'A. stabilisce il teorema di esistenza ed unicità di una soluzione generalizzata (u_1, u_2), assolutamente continua con le sue derivate prime rispetto alle singole variabili x_i, per quasi tutte le $(n-1)$-ple $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$, dotata di derivate seconde di quadrato sommabile, e verificante quasi ovunque la (1) e le (2). Quanto alle(3) esse sono verificate nel senso che le tracce di u_1 ed u_2 su Σ sono eguali mentre, detta $S(x_0, \rho)$ l'ipersfera di centro x_0 e raggio ρ e posto $S(x_0, \rho)\cdot D_s=S_s(x_0, \rho)$ riesce:

$$\lim_{\rho\to 0}\left(\int_{S_1(x_0,\rho)}\frac{du_1}{dv_1}d\sigma+\int_{S_2(x_0,\rho)}\frac{du_2}{dv_2}d\sigma\right)=0$$

per ogni $x_0 \in \Sigma-\Sigma\cdot\mathcal{F}D$.

Nell'ultima parte del lavoro i risultati ottenuti vengono perfezionati in quanto si studia il tipo di regolarità della soluzione, in dipendenza della classe dei coefficienti e dei dati al contorno, e si precisa come, di conseguenza, le condizioni al contorno e qualle di raccordo sono verificate.

C. Miranda (Napoli).

Temple, G. Generalized functions and Dirichlet's principle. Proc. Roy. Soc. London. Ser. A. **235** (1956), 444–453.

L'auteur décrit la méthode de résolution du problème de Dirichlet par le théorème de projection, utilisant les distributions [cas particulier des méthodes de J. L. Lions, Acta Math. **94** (1955), 13–153; MR **17**, 745].

L. Schwartz (Paris).

Foiaş, Ciprian; Gussi, Gheorghe; et Poenaru, Valentin. Une méthode directe dans l'etude du problème de Cauchy pour les équations aux dérivées partielles, hyperboliques, du second ordre, à deux variables. Rev. Math. Pures Appl. **1** (1956), no. 2, 61–98.

The authors reexamine the initial value problem for $\square u=f(x, y, u, u_x, u_y)$, where $\square u\equiv u_{xx}-u_{yy}$, by a method comparable to that of Cauchy-Peano for $y'=f(x, y)$. They consider systematically a more general problem which replaces $\square u$ by a limit of second differences, which may exist when u_{xx} and u_{yy} do not. These preliminary ideas go back to P. Montel [Ann. Sci. Ecole Norm. Sup. (3) **24** (1907), 233–334, Ch. II].

Paralleling the developments for $y'=f(x, y)$ they arrive at local existence theorems, uniqueness theorems comparable to L. Tonelli [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) **1** (1925), 272–277], a criterion for continuation comparable to A. Wintner [Amer. J. Math. **67** (1945), 227–284; MR **6**, 225] and some results regarding the continuous dependence of u upon the initial data and upon f.

R. W. McKelvey (Boulder, Colo.).

Baratta, Maria Antonietta. Sopra un problema cilindrico non lineare di propagazione del calore. Riv. Mat. Univ. Parma **6** (1955), 389–398.

Let R_1 and R_2 be the surfaces of two coaxial cylinders of radii a, b $(0<a<b)$. S_1 will be used to denote the domain interior to R_1 while S_2 will denote the domain between R_1 and R_2. Use is made of the Laplace transform to treat the following boundary value problem. Find two functions $U^{(i)}=U^{(i)}(r, t)$, $i=1, 2$, which satisfy the conditions:

$$U_{rr}^{(i)}+r^{-1}U_r^{(i)}=U_t^{(i)} \text{ in } S_i \text{ for } t>0;$$

$$(*)\qquad U^{(i)}(r, 0)=0 \text{ in } S_i;\ U_r^{(2)}(b, t)=C,\ t>0;$$

$$k_iU_r^{(i)}(a, t)=C_iT(t)+G(\Phi(t)),\ t>0.$$

Here C, C_i, k_i are constants, T and G are known functions of their argument, and $\Phi(t)=U^{(2)}(a+, t)-U^{(1)}(a-, t)$. Since the boundary conditions involve the difference of the unknown functions $U^{(1)}$ and $U^{(2)}$ on the surface R_1 there is a question whether there exists a solution of the system (*). As in an earlier paper the author [Boll. Un. Mat. Ital. (3) **11** (1956), 427–431; MR **18**, 358] shows that (*) will have a solution if $\Phi(t)$ satisfies a nonlinear Volterra integral equation of the type

$$\Phi(t)=\mu(t)+\int_0^t\frac{G(\Phi(\tau))}{\sqrt{(\pi(t-\tau))}}\sigma(t-\tau)d\tau.$$

The functions μ and σ are independent of Φ.

F. G. Dressel (Durham, N.C.).

Fage, M. K. Solution of the Cauchy problem by increasing the number of independent variables. Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 1022–1025. (Russian)

Consider

$$(1)\qquad q_n(w)\frac{\partial^nF}{\partial w^n}+p_n(x)\frac{\partial^nF}{\partial x^n}+\sum_{k=1}^{n-1}q_k(w, x)\frac{\partial^kF}{\partial w^k}+\sum_{k=0}^{n-1}p_k(w, x)\frac{\partial^kF}{\partial x^k}=H(w, x),$$

where $w=u+iv$ varies in a domain G, x real varies in $\Delta=[a, b)$, $-\infty<a<b\leqq+\infty$; q_k, p_k, H are continuous in the pair (w, x), one-valued and analytic in w, $q_n\neq 0$, $p_n>0$; by making a change of variables, it may be assumed that $p_n=1$, $q_n=(-1)^{n-1}$. Let $P_0=(w_0, x_0)$ be any point, $x_0>0$, and $\mathfrak{P}_{P_0}$ be the regular pyramid with vertex P_0, whose edges form 45° with the w-plane and whose base is a regular n-gon W_{P_0}. A point P_0 is called accessible if $W_{P_0}\subset G$; the set of accessible points is denoted by $\mathfrak{C}_0$. Theorem A: Let $f_i(w)$, $i=0, 1, \cdots, n-1$, be given regular functions in G; then there is a unique solution of (1) such that F and $\partial^kF/\partial x^k$, $k=1, \cdots, n$, are continuous in $\mathfrak{C}_0\cup G$, regular in the intersection of $\mathfrak{C}_0$ with any plane $x=$const and such that $\partial^iF(w, 0)/\partial x^i=f_i(w)$, $i=0, 1, \cdots, n-1$. Theorem B: The value of F at P_0 may be calculated by means of integrals depending on the values taken by the f_i and their derivatives in W_{P_0} (Riemann formula). The proofs are based on the transition to n real variables, the (in general not one-one) change of variables being

$$w=w_0-\sum_{i=1}^n t_i\varepsilon_i,\quad x=x_0-\sum_{i=1}^n t_i,$$

ε_i being the nth roots of unity; this leads to a type of equations considered before by the author [same Dokl. 108 (1956), 780–783; MR **18**, 214].

J. L. Massera.

Browder, Felix E. Parabolic systems of differential equations with time-dependent coefficients. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 914–917.

Soit $D=A(t)+\partial B(t)/\partial t$ un opérateur différentiel dans un ouvert G de $R_x{}^n\times R_t$, $A(t)$ et $B(t)$ étant des opérateurs différentiels en x, d'ordre $2m$ et $2s\leqq 2m-2$. L'A. donne d'abord un théorème d'existence des solutions faibles de l'équation $Du=T$ dans G. Il y a un lapsus p. 915: C n'est pas compact. Le th. 1, avec B positif, résulte de Lions,

C. R. Acad. Sci. Paris **242** (1956), 3028–3030 [MR **18**, 130].

Dans le cas $B(t)=1$, l'A. donne une méthode élégante pour montrer la régularité des solutions de $Du=T$.

Sous des hypothèses convenables sur la frontière de G, l'A. montre l'existence et l'unicité du „premier" problème aux limites pour D dans G; la méthode consiste à transformer localement G en un ouvert cylindrique et appliquer le théorème de Hille-Yosida. (Autre méthode dans Lions, loc. cit).
J. L. Lions (Lawrence, Kan.).

Caĭ, I. P. Special solutions of the equation of Lamé in elliptic coordinates. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **16** (1955), 113–120. (Russian)

Tanimura, Masayoshi. On the solution of some mixed boundary problems. II. Factorization of the kernel by contour integrals. Tech. Rep. Osaka Univ. **5** (1955), 337–348.

Part I was reviewed in MR **17**, 629. The method described there is now extended to plates and cylinders of infinite cross section.

Tanimura, Masayoshi. On the solution of some mixed boundary problems. III. Problems of a partial interval for symmetrical kernels. Tech. Rep. Osaka Univ. **6** (1956), 63–74.

For parts I and II see above. In this part (see author's summary) a solution is presented for the problem of finding two functions $U(x)$ and $W(x)$ whose Laplace transforms $\Psi(q)$ and $\Phi(q)$ are related by an equation $\Psi(q)=k(q)\Phi(q)$ with a given symmetric kernel $k(q)$ analytic at the origin, when one of them is given in an interval $|x| < a > 0$ and the other outside that interval.

See also: Seidenberg, p. 558; Wasow, p. 568; Albertoni, p. 570; Penzlin, p. 584; Friedman, p. 584; Citlanadze, p. 586; Nehari, p. 602; Misztal, p. 603; Manfredi, p. 603; Savin, p. 609; Sokolov, p. 609; Teodorescu, p. 613; van der Linden, p. 614; Crank, p. 616; Wolska, p. 618.

Difference Equations, Functional Equations

Verblunsky, S. On a class of differential-difference equations. Proc. London Math. Soc. (3) **6** (1956), 355–365.

The author considers the differential-difference equation

$$\sum_{p=0}^{m}\sum_{q=0}^{n} k_{pq}f^{(p)}(t+h_q)=g(t),$$

where $0=h_0<h_1<\cdots<h_n$, the complex constants k_{pq} satisfy the conditions $\sum_{p=0}^{m}|k_{pq}|^2>0$, $q=0, \cdots, n$ and $\sum_{q=0}^{n}|k_{mq}|^2>0$, and $g(t)$ is given continuous and of bounded variation in every finite interval. A solution of this equation is defined for all values of t by its values on any interval of length h_n, say $(0, h_n)$. Let

$$F(z)=\sum_{p=0}^{m}\sum_{q=0}^{n} k_{pq}z^{p}e^{zh_q}$$

and let the zeros of $F(z)$, all supposed simple, be enumerated in a sequence with nondecreasing absolute value. The author proves the following theorems: 1) A solution $f(t)$ continuous and of bounded variation on any finite interval and with m continuous derivatives is represented (with specific exceptions) by a series, uniformly convergent on every finite interval, of the form $\sum_{j=1}^{\infty} u_j(t)$ where $u_j(t)=\sum_{\nu=n_j+1}^{n_{j+1}} c_\nu(t)e^{z_\nu t}$ and the $c_\nu(t)$ are defined explicitly in terms of $g(t)$, $f(t)$ on $(0, h_n)$, the z_ν's and the k_{pq}'s. 2) If the set of distances between pairs of zeros of $F(z)$ has a positive greatest lower bound, the series $\sum_{\infty}^{\nu=1} c_\nu(t)e^{z_\nu t}$ converges uniformly on any finite interval. 3) If $f(t)$ is continuous and of bounded variation with m derivatives of bounded variation on $(0, h_n)$, there exists a series representation for $f(t)$ similar to that mentioned under 1) which converges boundedly to $f(t)$ in the open interval $(0, h_n)$. References are given to earlier works of Hilb, Leont'ev, Wright, and Langer. *P. E. Guenther* (Cleveland, Ohio).

Ghermanescu, Michel. Equations fonctionnelles linéaires à argument fonctionnel n-périodique. C. R. Acad. Sci. Paris **243** (1956), 1593–1595.

The author states two theorems on the existence of solutions of functional equations

$$a_0f+a_1f^1+\cdots+a_{n-1}f^{n-1}=0 \text{ or } g(M),$$

where $f^k(M)$ is defined as $f(\theta_k(M))$; here g and θ are given functions, $\theta_k(M)=\theta(\theta_{k-1}(M))$, and $\theta_n(M)=M$.
R. P. Boas, Jr. (Evanston, Ill.).

Ghermanescu, Michel. Equations fonctionnelles linéaires à argument fonctionnel n-périodique. C. R. Acad. Sci. Paris **244** (1957), 543–544.

Continuation of the paper reviewed above. The author states six more theorems on the existence and properties of solutions of linear functional equations.
R. P. Boas, Jr. (Evanston, Ill.).

Integral and Integrodifferential Equations

Adonc, M. T. Application of the method of degenerate kernels to non-linear integrodifferential equations. Akad. Nauk Azerbaĭdžan. SSR. Dokl. **11** (1955), 833–838. (Russian. Azerbaijani summary)

The author considers non-linear integrodifferential equations

$$(1)\qquad U(x)=\lambda\int_0^1 \Gamma[x, y, U(y), U'(y), \cdots, U^{(n)}(y)]dy,$$

where the integrand can be expressed in the form

$$\Gamma[x, y, U(y), U'(y), \cdots, U^{(n)}(y)] =\int_0^1 K(x, t)f(t, y, U(y), U'(y), \cdots, U^{(n)}(y))dt.$$

Under appropriate smoothness conditions on the kernel $K(s, t)$ and the natural continuity and Lipschitz conditions on the function f, the author shows that (1) has a unique solution when $|\lambda|$ is sufficiently small.

The method used is to approximate to K by partial sums of the expansion

$$K(s, t)=\sum_{i=1}^{\infty}\frac{\varphi_i(s)\psi_i(t)}{\lambda_i}$$

in terms of singular functions and singular values of K, to solve (1) when K is replaced by one of the resulting degenerate kernels, and then use an equicontinuity argument to prove the existence of a sequence of functions coverging to a solution of (1). Uniqueness is derived from the Lipschitz conditions by the usual type of argument.
F. Smithies (Cambridge, England).

Robinson, Lewis Bayard. A functional equation with double integrals. Rev. Ci., Lima **57** (1955), 94–99.

The author, extending an earlier work [Math. Mag. **23** (1950), 183–188; MR **12**, 416], considers the functional equation

$$\text{(I)}\qquad u(x,\ y)=f(x,\ y)+\lambda\int_1^{x^*}\int_1^{y^*} Nu(x_1,\ y_1)dY_1dX_1,$$

where

$$N=N\left(\frac{1}{1+x},\ \frac{1}{1+y};\ \frac{1}{1+x_1},\ \frac{1}{1+y_1}\right),$$

$$dX_1=d\left(\frac{1}{1+x_1}\right),\ dY_1=d\left(\frac{1}{1+y_1}\right),$$

and with f and N finite at infinity. It is stated that (I) can be solved by the Fredholm process for x, y real, after which one can pass to the complex domain by a successive approximation method. *I. M. Sheffer.*

★**Korobeinikov, B. P. On the integral equations of unsteady adiabatic gas motion.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 6 pp.

Translated from Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 509–512. The original Russian article was reviewed in MR **17**, 1150.

See also: Albertoni, p. 570; Baratta, p. 580; Mikusiński, p. 585.

Calculus of Variations

Bellman, Richard. On a class of variational problems. Quart. Appl. Math. **14** (1957), 353–359.

The paper is concerned with minimization of functionals of the form,

$$J(x)=\int_0^T\sum_{i=0}^{K} F_i[x^{(i-1)}(t)-b_i(t),\ t]dt,$$

over all functions $x(t)$ satisfying given initial conditions $x^{(i)}(t)=c_i$ $(i=0, \cdots, K-1)$. Such problems when considering engineering control problems or in inventory studies.

The general solution for the discrete analogue of this problem is sketched quickly making use of a recurrence relation. The remainder of the paper considers in more detail cases where the functions F_i have the form, $a_i(t)[x^{(i-1)}-b_i(t)]^2$. In the discrete case (with integrals replaced by sums and derivatives by differences of corresponding orders), the solution can be found by a remarkably simple finite algorithm. In the continuous case, the principle of optimality leads finally to solution of a system of ordinary nonlinear differential equations. *K. J. Arrow* (Palo Alto, Calif.).

See also: Stampacchia, p. 579; Adams, p. 603. Berk, 623.

TOPOLOGICAL ALGEBRAIC STRUCTURES

Topological Groups

Gluškov, V. M. Locally bicompact groups with a minimal condition for closed subgroups. Ukrain. Mat. Ž. **8** (1956), 135–139. (Russian)

Černikov [Mat. Sb. N.S. **7(49)** (1940), 35–64, 539–548; MR **2**, 5, 126] has investigated abstract locally solvable groups (that is, groups such that every finitely generated subgroup is solvable) with a minimal condition. The author generalizes his results by considering locally compact groups G with a minimal condition for closed subgroups. He shows that: (Th. 2) Such a group G is an extension of a compact Lie group with a discrete factor group; (Th. 3) If G is locally solvable, it is an extension of a finite product of 1-dimensional tori and discrete quasi-cyclic groups (groups of type p^∞) with a finite solvable factor group; (Th. 4) If G is locally nilpotent, it has a finite ascending central series and is an extension of a finite product of 1-dimensional tori and discrete quasi-cyclic groups with a finite nilpotent factor group; all toroidal subgroups belong to the center of G.

J. L. Tits (Brussels).

Shiga, Kôji. Representations of a compact group on a Banach space. J. Math. Soc. Japan **7** (1955), 224–248.

The author treats the connections between six different regularity conditions on a bounded representation of a compact group by linear operators on a Banach space, ranging from no condition at all to strong continuity. In the latter case the representation is completely decomposable in a certain sense into finite-dimensional ones. A typical special case of the main result is that a weakly continuous representation induces a strongly continuous representation in the second conjugate of the representation space in the canonical manner. For unitary representations on a Hilbert space or for separable groups the author's results are known to hold, more generally, for locally compact groups. A new proof of the complete reducibility of a continuous unitary representation on Hilbert space is given. *I. E. Segal* (Chicago, Ill.).

Tomita, Minoru. Harmonic analysis on locally compact groups. Math. J. Okayama Univ. **5** (1956), 133–193.

Functions on a separable locally compact group G are analyzed in a generally somewhat familiar manner in terms of a dual space of elementary positive definite functions on G. The Fourier transform of a function on G is defined as the function on E corresponding to it in the obvious manner. To some extent it is possible to transform in the reverse direction. However, the theory is complicated by the fact that there is no unique measure on E associated with this type of harmonic analysis, and in fact integration over E requires special study because in general E is merely completely regular.

For an abelian group, E can be identified with the character group, and the author's results specialize to variants of the well-known extensions of the Plancherel and Bochner theorems. For a compact group, E contains for each irreducible unitary representation equivalence class of dimension r a subset homeomorphic with complex projective r-space, so that the corresponding specializations of the author's results are less readily stated.

In part the paper consists of reformulations of essentially well-known material dealing with the pure state space of operator algebras, maximal abelian operator algebras, decomposition theory, etc. The proof in connection with Theorem 18 that a right convolution on $L_2(G)$ be a self-adjoint element of $L_2(G)$ is a self-adjoint operator glosses over the difficulty. *I. E. Segal.*

Tsuji, Kazô. Harmonic analysis on locally compact groups. Bull. Kyushu Inst. Tech. (Math., Nat. Sci.) no. 2 (1956), 16–32.

The main results of this paper resemble those of the paper reviewed above. The Fourier transform of a function on a locally compact group G is the naturally corresponding function on the space E of elementary positive definite functions, and the main point is to extend the theorems of Bochner and Plancherel in terms of this definition. For Bochner's theorem, this type of extension is well-known for separable groups, and while the author asserts it for inseparable groups, his proof depends on the use of Tomita's reduction theory, in which the proof of elementarity almost everywhere tacitly assumes the measurability of a certain function. The extension of the Plancherel theorem applies to non-unimodular groups, unlike the extension that depends on the factorial decomposition of the ring of operators determined by the group, which requires unimodularity. Another difference is that the former extension does not effectively reduce to the Peter-Weyl theorem on specialization to the case of a compact group. However the author's approach could be specialized to a fairly short treatment of the abelian case.

I. E. Segal (Chicago, Ill.).

Eberlein, W. F. The point spectrum of weakly almost periodic functions. Michigan Math. J. 3 (1955–1956), 137–139.

A weakly almost periodic (w.a.p.) function x on a locally compact abelian group b determines its formal Fourier series $\sum_{\lambda\in G^*} a(\lambda)(t, \lambda)$, where $a(\lambda)=M_s[x(s)(-s, \lambda)]$. It is shown that this Fourier series is the Fourier series of an almost periodic function on G and thus every w.a.p. function x on b is uniquely expressible as a sum x_1+x_2 of an almost periodic function x_1 and a function x_2 with $M(|x_2|^2)=0$. Also, for an w.a.p. function x, a character λ is in the uniform span of the translates of x if and only if $a(\lambda)\neq 0$. *N. Dunford* (New Haven, Conn.).

See also: Ramanathan, p. 557; Shiga, p. 582; Butzer, p. 585; Freudenthal, p. 591; Shibata, p. 627.

Lie Groups and Algebras

★ **Chevalley, Claude. Theory of Lie groups. I.** Princeton University Press, Princeton, N. J., 1946. (Third printing 1957). xi+217 pp. $2.75.

This book was reviewed in MR 7, 412.

Morimoto, Akihiko. Structures complexes invariantes sur les groupes de Lie semi-simples. C. R. Acad. Sci. Paris 242 (1956), 1101–1103.

Let G be a connected even-dimensional real Lie group, and let the Lie algebra $\mathfrak{g}$ of G be reductive, i.e. the direct sum of a semi-simple and an abelian algebra. Then, there exists infinitely many left-invariant complex analytic structures on G, such that the group of all right translations which are analytic automorphisms has a Cartan subalgebra of $\mathfrak{g}$ as Lie algebra. In the non-abelian case, there can be other left- but not right-invariant complex structures on G than those; for instance, GL($2n, R$) has left-invariant complex structures such that the group of all right translations which are analytic is isomorphic with GL(n, C). *J. L. Tits* (Brussels).

Dieudonné, Jean. On a theorem of Lazard. Amer. J. Math. 78 (1956), 675–676.

Let G be a formal Lie group of dimension one over a field K of characteristic $p>0$. In a previous paper [same J. 77 (1955), 218–244; MR 16, 789] the author, among other results, used his theory of hyperalgebras to prove that G is abelian except in a special case. This note completes the proof by treating that case. M. Lazard [C. R. Acad. Sci. Paris 239 (1954), 942–945; MR 16, 219] has shown, by a direct method, the abelian property of G when K is a commutative ring with an identity and without nilpotent element. *H. C. Wang* (New York, N.Y.).

Karpelevič, F. I. On the stratification of homogeneous spaces. Uspehi Mat. Nauk (N.S.) 11 (1956), no. 3(69), 131–138. (Russian)

The author proves that a homogeneous space G/H, where G and H are both semi-simple connected Lie groups may be homogeneously fibered by euclidean fibers over a base K/P, where K and P are maximal compact subgroups of G and H respectively. The concept of homogeneous fibering as defined by the author is essentially the covariant fibering introduced by Mostow in a similar situation. {It seems to the reviewer that the author's result is contained in a more general result of Mostow [Amer. J. Math. 77 (1955), 247–278; MR 16, 795]}.

W. T. van Est (Leiden).

Drinfel'd, G. I. On measure in Lie groups. Har'kov. Gos. Univ. Uč. Zap. 29=Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) 21 (1949), 47–57. (Russian)

The author establishes a necessary and sufficient condition in order that a Lie group G operating in a manifold M of dimension n leaves invariant a differential n-form in M. In case $M=G/H$, H connected Lie subgroup, this is precisely the well-known condition that H acts as a unimodular linear group in the tangent space to $M=G/H$ at the point H. In case $M=G/H$ is the euclidean plane, the existence of an invariant 2-form is investigated for various G's. *W. T. van Est* (Leiden).

Jennings, S. A.; and Ree, Rimhak. On a family of Lie algebras of characteristic p. Trans. Amer. Math. Soc. 84 (1957), 192–207.

Let G be a direct sum of n cyclic groups of prime order p, and let A be the group algebra of G over an algebraically closed field Φ of characteristic p. Let $D_0, D_1, \cdots, D_m$ be commuting derivations of A (operating on the left) such that: i) $\sum f_i D_i=0$, $f_i\in A$, implies all $f_i=0$; ii) $D_i f=0$ for all i implies $f\in\Phi$; iii) $D_i f=\lambda_i f$ for all i ($\lambda_i\in\Phi$) implies either $f=0$ or f is a unit in A. Let $a_0, a_1, \cdots, a_m\in A$ be such that $D_i a_j=D_j a_i$ for all i, j, and let $L(D_i, a_i)$ be the set of all derivations of A of the form $\sum f_i D_i$, where $\sum D_i f_i=\sum a_i f_i$. Each $L(D_i, a_i)$ is a Lie algebra, and the authors study the Lie algebras so obtained. These algebras generalize several classes considered by Albert and Frank [Univ. e Politec. Torino. Rend. Sem. Mat. 14 (1954–55), 117–139; MR 18, 52]. In every case, the D_i can be chosen to be a principal system, i.e., $D_i f\in\Phi$ for all i implies $f\in\Phi$. This is assumed in the sequel. The algebra $L(D_i, a_i)$ is said to be of type I if there is a non-zero $b\in A$ such that $D_i b=a_i b$ for all i; otherwise L is of type II. This property is independent of the principal system used to define it. Let F_c be the family of those $L(D_i, a_i)$ for which all $a_i\in\Phi$, F_0 the family of those of the form $L(D_i, 0)$, upon suitable choice of $\{D_i\}$. If

$L \in F_0$, its derived algebra L' is simple of dimension $m(p^n-1)$, $1 \leq m < n$. If $F_{\rm I}$, $F_{\rm II}$ respectively, denote the classes of types I, II, if $1<m<n$, and if $L \in F_e \cap (F_{\rm I}-F_0)$, then L' is again simple of dimension $m(p^n-1)$. If $m=1$, $n>1$, $p>2$, $L \in F_e \cap (F_{\rm I}-F_0)$, then L'' is simple of dimension p^n-2. If $1<m<n$, or if $1=m<n$ and $p>2$, each $L \in F_e \cap F_{\rm II}$ is simple of dimension mp^n. A brief discussion is included of the relation of the simple algebras so obtained to those of Albert and Frank. *G. Seligman.*

See also: Baxter, p. 557; Amitsur, p. 557; Freudenthal, p. 591; Ishihara, p. 599; Obata, p. 599.

Topological Vector Spaces

Shiga, Kôji. Bounded representations on a topological vector space and weak almost periodicity. Jap. J. Math. **25** (1955), 21–35 (1956).

The author, observing that the theory of almost-periodic (AP) complex-valued functions on a group G can be interpreted as a theory of bounded representations of G, presents an analogous interpretation of the Bochner-v. Neumann theory of AP functions with values in a convex topological vector space L. (However, v. Neumann's countability restriction for L is first abandoned.) A number of propositions are shown to be equivalent (and eventually to be true). One of these is as follows. Let L be as above, with the property that the closed-convex hull of each totally bounded set is compact. Let G act in L in such a way that orbits are totally bounded. Then L is completely decomposable. Another theorem gives sufficient conditions that L should have "sufficiently many" composition series. A composition series is a system of closed subspaces invariant under G, well-ordered by inclusion (ending in (0)) where the quotients L_n/L_{n+1} are finite-dimensional. *R. Arens* (Los Angeles, Calif.).

Henstock, R. Linear and bilinear functions with domain contained in a real countably infinite dimensional space. Proc. London Math. Soc. (3) **6** (1956), 481–500.

Let (m) be the space of bounded real sequences, and Q the ball consisting of those $x=\{x_n\}$ with $\|x\| \leq \frac{1}{2}$. Regarding Q as the Cartesian product of the intervals $[-\frac{1}{2}, \frac{1}{2}]$, give it the usual product measure. In a preceding paper [Proc. London Math. Soc. (3) **5** (1955), 238–256; MR **17**, 176], the author obtained the following results. Let J be a linear functional with domain $M \subset (m)$. Then, if $M \cap Q$ has positive measure, its measure is 1, and J may be defined a.e. in Q by $J(x)=\sum c_k x_k$, where $\sum c_k^2 < \infty$. Conversely, if $\sum c_k^2 < \infty$, then $\sum c_k x_k$ converges for a.e. x in Q. The object of the present paper is to extend these considerations to bilinear functions. Let $A(x, y)$ be a bilinear function defined on $M \times M$, and positive semi-definite. Then, if $M \cap Q$ has positive measure, there is a matrix $[a_{nk}]$ such that $\sum |a_{mn}| < \infty$ and such that for a.e. x in M, $A(x, x)=\sum a_{nk}x_n x_k$. Conversely, if $[a_{nk}]$ obeys $a_{nk} \geq 0$, $(a_{nk})^2 \leq a_{nn}a_{kk}$, $a_{nk}=a_{nk}$, then $\sum a_{nk}x_n x_k$ converges for a.e. x in M. Similar results are obtained in which the behavior of A and J are studied on the hyperplane $H=M+y$. If $A(x+y, x+y)$ is defined for a.e. x in Q, then it is defined for $x=0$ and

$$A(x+y, x+y)=\sum a_{nk}x_n x_k+\sum c_n x_n+A(y, y),$$

where $[a_{nk}]$ and c_n obey appropriate conditions. {These results have close connections with certain classical theorems in the theory of probability. In particular, theorem 4 could have been established in a sharper form by appeal to the law of the iterated logarithm.}
R. C. Buck (Madison, Wis.).

Penzlin, Fritz. Distributionentheoretische Behandlung von Anfangswertproblemen relativistischer Wellengleichungen. Wiss. Z. Friedrich-Schiller-Univ. Jena **5** (1955/56), 137–149.

Exposé résumé de la théorie des distributions, d'après la méthode de H. König [Math. Nach. **9** (1953), 129–148; MR **14**, 1072]. Convolution, intégration et dérivation d'ordre non entier, distributions de Riesz pour les équations d'ondes, équation de Klein-Gordon, problèmes de Cauchy, équation des champs en mécanique quantique.
L. Schwartz (Paris).

Ehrenpreis, Leon. On the theory of kernels of Schwartz. Proc. Amer. Math. Soc. **7** (1956), 713–718.

Le „théorème des noyaux" [Schwartz, Proc. Internat. Congress Math., Cambridge, Mass., 1950, v. 1, Amer. Math. Soc., Providence, R.I., 1952, pp. 220–230; J. Analyse Math. **4** (1954/55), 88–148, voir p. 143; MR **13**, 562; **18**, 220] exprime qu'il y a isomorphisme topologique entre l'espace $\mathscr{D}_{x,y}'$ des distributions sur $X^l \times Y^m$ et l'espace $J=\mathscr{L}(\mathscr{D}_y; \mathscr{D}_x')$ des applications linéaires continues de $\mathscr{D}_y$ dans $\mathscr{D}_x'$, muni de la topologie de la convergence bornée. L'auteur donne ici de ce théorème une démonstration très élémentaire. On a trivialement $\mathscr{D}_{x,y}' \subset J$; ceci exprime que tout noyau $T_{x,y} \in \mathscr{D}_{x,y}'$ définit une application linéaire continue $v \to \int_{Y^m} T_{x,y}v(y)dy$ de $\mathscr{D}_y$ dans $\mathscr{D}_x'$; d'autre part l'injection de $\mathscr{D}_{x,y}'$ dans J est trivialement continue. La topologie de $\mathscr{D}_{x,y}'$ est celle de la convergence uniforme sur les parties bornées de $\mathscr{D}_{x,y}$; celle de J est identique à celle de la convergence uniforme sur les produits tensoriels $A_x \otimes A_y$ de parties bornées de $\mathscr{D}_x$ et $\mathscr{D}_y$ respectivement. Mais le développement en série de Fourier $\varphi(x, y)=\sum_{n,p} a_{n,p} \exp(2i\pi(nx+py))$ des fonctions périodique de x, y, permet de montrer que toute partie bornée de $\mathscr{D}_{x,y}$ est contenue dans l'enveloppe convexe fermée d'un produit $A_x \otimes A_y$. Donc la topologie de $\mathscr{D}_{x,y}'$ est identique à la topologie induite par J. Mais $\mathscr{D}_{x,y}'$ est complet, donc il est fermé dans J. D'autre part, si $T \to T * \rho$ est une régularisation ($\rho \in \mathscr{D}_x$), l'application $v \to L(v) * \rho$ est continue de $\mathscr{D}_y$ dans $\mathscr{E}_x$, quelle que soit $L \in J$; et quand ρ tend vers δ dans $\mathscr{E}_x'$, cette application tend vers L dans J; comme toute application linéaire continue de $\mathscr{D}_y$ dans $\mathscr{E}_x$ se représente aisément par un noyau, toute $L \in J$ est adhérente à $\mathscr{D}_{x,y}'$, donc $\mathscr{D}_{x,y}'$ est dense dans J et fermé, et par suite identique à J.
L. Schwartz (Paris).

★ **Friedman, B. An abstract formulation of the method of separation of variables.** Proceedings of the conference on differential equations (dedicated to A. Weinstein), pp. 209–226. University of Maryland Book Store, College Park, Md., 1956.

Let $A_1, \cdots, A_R$ and $B_1, \cdots, B_R$ be self-adjoint operators in the Hilbert spaces H_1 and H_2 respectively and let L be a self adjoint operator in the product $H_1 \otimes H_2$ which has the form $L=A_1 \otimes B_1+\cdots+A_R \otimes B_R$. Let the operators $A_1, \cdots, A_R$ have a common spectral representation, i.e., for some self adjoint spectral resolution E, there are real Borel functions $\alpha_1, \cdots, \alpha_R$ such that $A_j u=\int \alpha_j(\lambda)E(d\lambda)u$ for every u in the intersection of the domains of A_i. Conditions under which the operator L has an everywhere defined inverse are given and the inverse is explicitly represented in terms of an integral

involving E. In case $R=2$ and $L=A\otimes I_2+I_1\otimes B$ where I_1, I_2 are the identity operators in H_1, H_2 respectively, an alternate form is given which expresses the inverse of L by a contour integral involving the resolvents of A and $-B$. The results constitute an abstract formulation of a method for solving certain partial differential boundary value problems which is more general than the method of separation of variables. A number of interesting illustrations are given. *N. Dunford* (New Haven, Conn.).

Butzer, P. L. Halbgruppen von linearen Operatoren und eine Anwendung in der Approximationstheorie. J. Reine Angew. Math. **197** (1957), 112–120.

Let T_t ($t\geqq 0$) be a uniformly bounded strongly continuous one-parameter family of operators on a Banach space with $T_0=I$. The author treats the approximation of T_tx by t-dependent linear combinations of the $T_{i/n}x$, for $0\leqq i\leqq n$. A fairly typical result is that

$$\lim_{n\to\infty}\left\{\sum_{i=0}^{n}\binom{n}{i}t^i(1-t)^{n-i}T_{i/n}\right\}x=T_tx$$

uniformly for $0\leqq t\leqq 1$, which the author points out can also be derived by the methods of Hille [Proc. Nat. Acad. Sci. U.S.A. **28** (1942), 421–424; MR **4**, 163], and which reduces for suitable specialization to a proof of the Weierstrass approximation theorem by means of the Bernstein polynomials, and thereby extends the list of proofs of the theorem by essentially semi-group methods. A number of more refined results, which however are not easily summarized, are obtained concerning the indicated approximation problem. *I. E. Segal* (Chicago, Ill.).

Butzer, Paul L. Sur la théorie des demi-groupes et classes de saturation de certaines intégrales singulières. C. R. Acad. Sci. Paris **243** (1956), 1473–1475.

The notion of a class of saturation related to a process of summation as introduced by Favard [Ann. Mat. Pura Appl. (4) **29** (1949), 259–291; MR **11**, 669] is extended to a semigroup of operators on a Banach space X. If A is the infinitesimal generator of the semigroup $\{T(\xi)\}$, $\xi\geqq 0$, then for x in the domain of A, $\|T(\xi)x-x\|\leqq\xi\|Ax\|$. If X is weakly complete a result of Hille [Functional analysis and semi-groups, Amer. Math. Soc. Colloq. Publ., v. 31, New York, 1948; MR **9**, 594] is complemented by showing that $\{T(\xi)\}$ is saturated and that the set of non-invariant x in the domain of A is the class of saturation of $\{T(\xi)\}$. Applications are stated to certain summability processes. *P. Civin* (Eugene, Ore.).

Mikusiński, J. Le calcul opérationnel d'intervalle fini. Studia Math. **15** (1956), 225–251.

Dans divers travaux antérieurs, l'auteur a étudié le corps des fractions de l'anneau de convolution des fonctions continues sur la demi-droite $t\geqq 0$ (bibliographie donnée dans l'article). Il remplace ici la demi-droite par l'intervalle fini $0\leqq t<T$. Soit C_T l'espace vectoriel des fonctions continues sur l'intervalle $0\leqq t<T$, considéré comme algèbre pour la convolution

$$fg=f*g,\ \text{avec}\ f*g(t)=\int_0^t f(\tau)g(t-\tau)d\tau.$$

C_T a des diviseurs de 0, à savoir les fonctions nulles au voisinage de $t=0$; soit $C_T{}^*$ l'ensemble des éléments qui ne sont pas diviseurs de 0. Alors A_T sera l'anneau des fractions de C_T; tout élément de A_T se représente comme quotient a/b, $a\in C_T$, $b\in C_T{}^*$. L'auteur étudie en détail l'algèbre A_T: diviseurs de 0, solutions d'équation algébriques, d'équations différentielles (dans un sens facile à préciser). Tout élément s'écrit sous la forme $h^\alpha a$, où a n'est pas diviseur de 0, et où h^α est la „masse unité" au point α, $0\leqq\alpha<T$. L'exponentielle $\exp(\lambda w)$ est définie comme la solution de l'équation différentielle $x'(\lambda)=wx(\lambda)$, prenant la valeur unité pour $\lambda=0$ (l'unité est ici la mesure de Dirac); si une telle solution existe, elle est unique; w est dit logarithme droit (resp. gauche, resp. bilatère) si $\exp(\lambda w)$ existe pour $\lambda\geqq 0$ (resp. pour $\lambda\leqq 0$, resp. pour λ quelconque).

Par exemple $-s^\alpha$ (s=dérivée de la mesure de Dirac) est logarithme bilatéral pour $\alpha<1$, logarithme droit pour $\alpha=1$, est n'est pas logarithme pour $\alpha>1$. L'auteur généralise de nombreux résultats de travaux antérieurs, et montre comment ce formalisme, de maniement très commode, a des applications à la théorie des équations aux dérivées partielles à coefficients constants; l'étude de l'exponentielle définie plus haut permet de discuter les problèmes d'existence, d'unicité, d'indépendance linéaire (au sens de la convolution). Signalons un lapsus: page 233, il est écrit que la série

$$1-\frac{s^\alpha}{1!}\lambda+\frac{s^{2\alpha}}{2!}\lambda^2+\cdots$$

converge vers $\exp(-s^\alpha\lambda)$ pour $\alpha<1$; c'est évidemment $\alpha\leqq 0$ qu'il faut lire. *L. Schwartz* (Paris).

Tsuji, Kazô. N^*-algebras and finite class groups. Bull. Kyushu Inst. Tech. (Math., Nat. Sci.) no. **1** (1955), 1–9.

An N^*-algebra is defined by the author as a complete normed $*$-algebra over the complex field with a center-valued trace. The main result is that for such an algebra with a unit there exists a canonical correspondence between (a) the maximal two-sided ideals, (b) the extreme numerical-valued traces, (c) the maximal ideals of the center. Applications are made to group algebras. *I. E. Segal* (Chicago, Ill.).

See also: Pták, p. 554; Feller, p. 575; Bellman, p. 576; Temple, p. 580; Rudin, p. 587; Bendat and Sherman, p. 588; Korányi, p. 588; Sz.-Nagy, p. 588; Chong, p. 588; Ogasawara and Maeda, p. 589; Maeda, p. 589; Sherman, p. 625.

Banach Spaces, Banach Algebras

Finkelstein, David. On relations between commutators. Comm. Pure Appl. Math. **8** (1955), 245–250.

Let A_1 be the free associative ring of any number of generators x, y, $\cdots$ over the field of real or complex numbers. The elements which can be derived from $x, y, \cdots$ by addition, multiplication by scalars and forming the Lie bracket of two elements represent the Lie subring L of A_1. The paper contains new characterizations of the elements of L implying in particular Friedrich's criterion first proved by Magnus [same Comm. **7** (1954), 649–673; MR **16**, 790]. The derivations employ a calculus in the algebra A_2 of transformations in A_1 generated from the left transformations $f\to fu$ and right transformations $f\to fu$, u being a fixed and f a variable element of A_1. Calling these transformation u and u' respectively and setting $u-u'=\Delta u$, the following formula is essential:

$$\Delta f(x,y,\cdots)=\frac{\partial f(x,y,\cdots)}{\partial x}\Delta x+\frac{\partial f(x,y,\cdots)}{\partial y}\Delta y+\cdots.$$

Here a derivative, e.g. $\partial f/\partial x$ represents the transformation which transforms an element f of A_1 into the element appearing as coefficient of ε in the expansion of

$$f(x+\varepsilon t, y, \cdots)$$

with respect to ε. The fundamental criterion for a Lie-element $f(x, y, \cdots)$ of A_1 can now be expressed in the following way: f is a Lie-element if and only if

$$\Delta f(x, y, \cdots)=f(\Delta x, \Delta y, \cdots).$$

Application of this formula leads to a representation of $\log e^x e^y$ bringing into evidence the Baker-Hausdorff theorem that it belongs to L. *C. Loewner.*

Schwartz, J. Two perturbation formulae. Comm. Pure Appl. Math. 8 (1955), 371–376.

The author proves the following theorem. If S, N are commuting operators, f analytic in D, including $\sigma(S)$, the spectrum of S, and every point within a distance of $\sigma(S)$ not greater than ε; if $\sigma(N)$ is within a distance less than ε from the origin, then f is analytic on $\sigma(S+N)$ and, $f(S+N)=\sum_{n=0}^{\infty} f^{(n)}(S)N^n/n!$, where the series converges in the uniform topology. The proof is straightforward. For noncommuting operators a corresponding formula is unknown. However the following is proved: Let f be analytic in D, containing $\sigma(T)$, T bounded. Then $f(S)$, according to the preceding is defined in a neighborhood U of T and $f(S)$ has a Fréchet derivative $df(T)/dT$. If $g(z_1, z_2)=(f(z_1)-f(z_2))/(z_1-z_2)$, then $df(T)/dT=g(rT, lT)$, where rT denotes right and lT left multiplication by T. This theorem is due to Finkelstein [see the paper reviewed above] but this is the first rigorous proof and interpretation of the formula. From

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[R(\xi;\varepsilon S+T)-R(\xi,T)]=R(\xi,T)SR(\xi,T),$$

$lR(\xi;T)=R(\xi;lT)$ and $rR(\xi;T)=R(\xi;rT)$ we get

$$d/d\varepsilon f(T+\varepsilon S)=(2\pi i)^{-1}\int_C f(\xi)R(\xi;lT)R(\xi,rT)Sd\xi$$

and hence

$$df(T)/dT=(2\pi i)^{-1}\int_C f(\xi)R(\xi;lT)R(\xi;rT)d\xi.$$

Taking C_1 surrounding $\sigma(T)$, but inside C, we get

$$R(\xi;lT)=(2\pi i)^{-1}\int_C(\xi-\xi_1)R(\xi_1;lT)d\xi_1$$

which helps to finish the proof. *František Wolf.*

Weiss, Guido. A note on Orlicz spaces. Portugal. Math. 15 (1956), 35–47.

The author gives a new approach to the theory of Orlicz spaces, starting from their introduction by means of a norm, and eliminating the need for the introduction of the conjugate in the sense of Young. It is shown that for any non-decreasing real-valued convex function ϕ defined on the positive real line and vanishing at 0, the closure under positive scalar multiplication of the class of all measurable functions f on the (arbitrary) measure space M such that $\int_M \phi(|f|)d\mu<\infty$, where μ is the measure on M, is a Banach space L_ϕ relative to an explicitly given norm. Conjugate spaces are treated briefly. The author points out how his approach may be adapted to give a natural generalization of the H_p spaces of holomorphic functions in the unit disc to H_ϕ spaces. *I. E. Segal.*

Citlanadze, È. S. On a certain class of non-linear integral equations. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze 22 (1956), 227–236. (Russian)

Let F be a functional on a Banach space E, $L=\text{grad } F$ the gradient of F, i.e. (Lx, h) is the strong differential coefficient of $F(x+th)$ at $t=0$, and let φ be a functional on E whose gradient is homogeneous. Then the existence of eigenvalues of $Lx=\lambda L\varphi x$ is demonstrated, under a rather complicated set of conditions, by a method which combines discussion of the categories of sets, in the Lyusternik-Schnirelman sense, with a method of approximation to proper vectors. *J. L. B. Cooper.*

Calderón, A. P.; and Zygmund, A. A note on the interpolation of sublinear operations. Amer. J. Math. 78 (1956), 282–288.

Let (α_1, β_1), (α_2, β_2) be two points of the square $0\leqq\alpha\leqq 1$, $0\leqq\beta\leqq 1$. Let R be a measure space and let T be simultaneously a bounded operator from L^{1/α_1} to L^{1/β_1} and from L^{1/α_2} to L^{1/β_2} defined on R and having the following properties, (a) if $f=f_1+f_2$ and Tf_1, Tf_2 are defined, then Tf is defined, (b) if Tf is defined $T(kf)$ is defined for any scalar k, (c) $\|Tf_1+Tf_2\|\leqq\|Tf_1\|+\|Tf_2\|$, (d) $\|T(kf)\|=|k|\cdot\|Tf\|$. Let $\|T\|_1=M_1$ for T considered as an operator from L^{1/α_1} to L^{1/β_1}, and $\|T\|_2=M_2$ for T considered as an operator from L^{1/α_2} to L^{1/β_2}. Then if $0\leqq t\leqq 1$, $\alpha=(1-t)\alpha_1+t\alpha_2$, $\beta=(1-t)\beta_1+t\beta_2$, T is also a bounded operator from $L^{1/\alpha}$ to $L^{1/\beta}$ with norm $M\leqq M_1^{(1-t)}M_2^t$. This is a generalization of a theorem of M. Riesz [Acta Math. 49 (1927), 465–497] who proved the result for bounded linear operators. *R. E. Fullerton.*

Foias, Ciprian. Elementi completamente continui e quasi completamente continui di un'algebra di Banach. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 20 (1956), 155–160.

The author defines completely continuous elements of a Banach algebra B by analogy with certain phenomena in the ring of operators over Hilbert space. Starting with elements $j\in B$ which have the property $(xjy)^2=f_j(yx)xjy$, for arbitrary $x, y\in B$, where f_j is linear over B, and which correspond to transformations in Hilbert space of rank 1, he considers the set K of finite sums $j_1+\cdots+j_n$ (transformations of finite rank), and the closure $\bar{K}$ of the latter (the set of completely continuous elements). For these he proves four theorems (the last one on quasi-completely continuous elements) of a function theoretic nature, analogous to certain expansions in the Fredholm theory. For example (writing $(e-\lambda x)^{-1}=e+\lambda Q(\lambda, x)$), if $x\in \bar{K}$ then $Q(\lambda, x)$ is rational and the coefficients of the principal parts at each pole belong to $\bar{K}$. $Q\in \bar{K}$ for each λ in the resolvent set of x. *E. R. Lorch.*

Civin, Paul; and Yood, Bertram. Regular Banach algebras with a countable space of maximal regular ideals. Proc. Amer. Math. Soc. 7 (1956), 1005–1010.

Let B be a complex commutative Banach algebra and let $\mathfrak{M}$ be its space of regular maximal ideals. Assume (i) B is regular and (ii) $\mathfrak{M}$ is countable. If A is a subalgebra of B, call A "non-determining" if the ring of functions on $\mathfrak{M}$ representing A under the Gelfand mapping is not dense in the ring of functions on $\mathfrak{M}$ representing B. The authors prove: Theorem. Let A be a proper closed subalgebra of B which is maximal, i.e. such that there exists no closed subalgebra of B lying properly between A and B. Assume that A is non-determining. Then either A is a maximal regular ideal of B or there exist distinct points

m_1 and m_2 in $\mathfrak{M}$ such that A consists of all x in B with $x(m_1)=x(m_2)$. Corollary. If S is a countable compact Hausdorff space and $C(S)$ is the algebra of all continuous complex functions on S, then $C(S)$ contains no maximal proper closed subalgebra which separates points on S and contains constants. The authors also prove the following: Theorem. Let B be as above. Then the set B_0 of functions $\hat{x}$ on $\mathfrak{M}$ with x in B which vanish outside some compact set of $\mathfrak{M}$ is dense in $C(\mathfrak{M})$. Furthermore, an ideal I of B is contained in a regular maximal ideal if and only if I is non-determining. *J. Wermer.*

Rudin, Walter. Subalgebras of spaces of continuous functions. Proc. Amer. Math. Soc. **7** (1956), 825–830.

Let X be a compact Hausdorff space and $C(X)$ the space of all complex-valued continuous functions on X. A proper closed subalgebra of $C(X)$ is maximal if no closed subalgebra of $C(X)$ lies properly between it and the whole space. Let K be the Cantor set. The author proves the existence of a maximal subalgebra in $C(K)$ which separates points on K and contains constants. If E is any compact plane set, let A_E denote the algebra of all continuous functions on E which can be extended to be continuous on the whole plane and analytic outside E, including ∞. For a certain totally disconnected set E, the author shows, by use of Zorn's Lemma, that A_E is contained in a maximal subalgebra of $C(E)$. It is not known whether A_E is itself maximal. Since the Cantor set is homeomorphic to E, the existence of a maximal subalgebra in $C(K)$ follows. If now X is an arbitrary compact space and X_0 is a closed subset of $C(X)$ and B_0 is a maximal subalgebra of $C(X_0)$, then the class of functions in $C(X)$ whose restriction to X_0 lies in B_0 forms a maximal subalgebra of $C(X)$. It follows from the preceding that if X is any space having a subset homeomorphic to K, then $C(X)$ contains a maximal subalgebra separating points on X. Let next J be a Jordan arc in the space C^n of n complex variables $z_1, \cdots, z_n$, and write $P(J)$ for the class of functions on J which are uniformly approximable on J by polynomials in the z_i. Walsh has shown that if $n=1$, $P(J)=C(J)$ for every arc J. The reviewer [Ann of Math. (2) **62** (1955), 269–270; MR **17**, 255] has exhibited an arc J in C^3 for which $P(J)\neq C(J)$, making use of an algebra A_E as defined above with E a Jordan arc in the plane having positive plane measure. The author closes the remaining gap by proving the existence of an arc J in C^2 with $P(J)\neq C(J)$. His construction is based on an algebra A_E with E totally disconnected which first yields a totally disconnected perfect set in C^2 on which not every continuous function is approximable by polynomials. The desired arc is then obtained by threading an arc through this totally disconnected set. In the positive direction, the author proves the following result on approximation by polynomials: Theorem. Let $\bar{K}$ be a compact set in the space of one complex and n real variables $z, t_1, \cdots, t_n$. Denote by $\bar{K}_t$ the set of z with (z, t) in $\bar{K}$, where $t=(t_1, \cdots, t_n)$. Assume $\bar{K}_t$ does not separate the z-plane for any t. Suppose f is in $C(\bar{K})$ and set $f_t(z)=f(z, t)$. If for each t, f_t is analytic at every interior point of $\bar{K}_t$, then f can be uniformly approximated on K by polynomials in z, $t_1, \cdots, t_n$. *J. Wermer* (Princeton, N.J.).

Harazov, D. F. On finding of eigenvalues and the approximation of solutions of functional equations in Banach spaces. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **22** (1956), 237–250. (Russian)

Let $H(\lambda)$ be an operator on a Banach space to itself, analytic and completely continuous in a circle in the plane of the complex variable λ. Let $G_n(\lambda)$ be a sequence of finite-dimensionally valued operators, approximating to $H(\lambda)$ uniformly in the circle and analytic in the circle: if the space has a countable basis, the G_n may be polynomials in λ. The relations between solutions of the equations (1) $x-H(\lambda)x=0$ and (2) $x-H(\lambda)x=y$, and equations (3) $x-G_n(\lambda)x=0$, (4) $x-G_n(\lambda)x=y_n$ are discussed. For a point λ_0 to be a proper value of (1) it is necessary and sufficient that it is a limit of a sequence of proper values of (3); if it is a regular value of (2) with solution x it must be a regular value of (4) for all large enough n, and x is a limit of the solutions of (4) with any sequence $y_n\to y$. In a space in which the unit sphere is weakly compact, if λ_n is a sequence of eigenvalues of (3) with corresponding eigenvectors x_n, and if $\lambda_n\to\lambda_0$, then $\{x_n\}$ has as limiting point an eigenvalue of (1) for the eigenvalue λ_0. Finally, the equations (3) and (4) are replaced by finite systems of simultaneous algebraic equations, and the relationships with the solutions of these are discussed; the bearing of this on the convergence of Galerkin's method is worked out. *J. L. B. Cooper* (Cardiff).

Krabbe, G. L. Abelian rings and spectra of operators on l_p. Proc. Amer. Math. Soc. **7** (1956), 783–790.

A correspondence is established between the set $\mathfrak{M}$ of bounded measurable complex valued functions on $[-\pi, \pi]$ and a set of infinite matrices in the following manner. The matrix $A_{\mathfrak{K}}$ corresponds to the function $A(\theta)$ if $A_{\mathfrak{K}}=(a_{n,\nu})=(a_{n-\nu})$, where the a_k are Fourier coefficiets in the exponential form of the Fourier series corresponding to $A(\theta)$. It is shown that for any $p>1$, $A_{\mathfrak{K}}$ is a bounded linear operator over l_p and in case the functions involved are in the ring $\mathfrak{B}$ of functions of bounded variation over $[-\pi, \pi]$, the correspondence is an algebraic isomorphism between $\mathfrak{B}$ and $\mathfrak{B}_{\mathfrak{K}}$ such that if $A(\theta)$ has an inverse in $\mathfrak{B}$ then $A_{\mathfrak{K}}^{-1}$ is a bounded linear operator on l_p. If the functions involved are continuous on $[-\pi, \pi]$ then $\sup_{-\pi\leq\theta\leq\pi} A(\theta)=\|A_{\mathfrak{K}}\|$. In this case the spectrum of $A_{\mathfrak{K}}$ is a connected subset of the image of $[-\pi, \pi]$ under A. If A is analytic $\sigma(A_{\mathfrak{K}})$ is the entire image of $[-\pi, \pi]$ under A and the point spectrum of $A_{\mathfrak{K}}$ is void except in the trivial case of a constant A. *R. E. Fullerton.*

Pi Calleja, Pedro. The normal-derivative numbers of vectorial functions. Rev. Un. Mat. Argentina **17** (1955), 161–172 (1956). (Spanish)

Let $f(x)$ be a function of the real variable x having values in a real normed linear space and define

$$N_+f(x)=\liminf_{y\to x+}\frac{\|f(y)-f(x)\|}{|y-x|}.$$

The following theorem is typical of those proven by the author. Let $f(x)$ be continuous on the interval $[a, b]$ and $g(x)$ a real valued function of the real variable x continuous and increasing on $[a, b]$. Suppose in $[a, b]$,

$$(*)\qquad N_+f(x)\leq D_+g(x)$$

with the possible exception of a denumerable set of points. Then $\|f(b)-f(a)\|\leq g(b)-g(a)$ if both members of (*) are not infinite. Similar results are obtained for $N^+f(x)$, $N_-f(x)$ and $N^-f(x)$ which are defined by analogy with $N_+f(x)$. These theorems are related to older results of Scheeffer [Acta Math. **5** (1884), 183–194, 279–296] and Dini [Fondamenti per la teoria delle funzioni di variabili reali, Nistri, Pisa, 1878]. *V. F. Cowling* (Lexington, Ky.).

See also: Shiga, p. 582; Hongo, p. 588.

Hilbert Space

Hongo, Eishi. A note on the commutor of certain operator algebras. Bull. Kyushu Inst. Tech. (Math., Nat. Sci.) no. **1** (1955), 19–22.

A left * algebra A is a complex associative algebra which has an inner product and a continuous * operation for which $(xy, z)=(y, x*z)$ for all x, y, z in A, the map $y\to xy$ is continuous, and the set of all xy is dense. This is a generalization of Dixmier's "quasi-unitary algebras". The "commutation theorem" is proven for A: If H is the completion of A, and L and R are the weak closures of the sets of bounded linear operators on H which are extensions of, respectively, left multiplications and right multiplications in A, then any bounded linear operator on H which commutes with all of L is in R.

J. Feldman (New York, N.Y.).

Bendat, Julius; and Sherman, Seymour. Monotone and convex operator functions. Trans. Amer. Math. Soc. **79** (1955), 58–71.

Let $f(x)$ be a real valued function of the real variable x defined in an interval (a, b) and X a real symmetric matrix of order n with all eigenvalues in (a, b). Then, by definition, $f(X)$ is the symmetric matrix of order n with the same eigenvectors as X, but real eigenvalue λ replaced by $f(\lambda)$. The function $f(x)$ is monotone of order n if $A\leqq B$ implies $f(A)\leqq f(B)$. An inequality $A\leqq B$ means that $B-A$ is the matrix of a non-negative quadratic form. The monotone functions of order n can be characterized by difference or differential inequalities [see Löwner, Math. Z. **38** (1934), 177–216; Dobsch, ibid. **43** (1937), 353–388]. In particular the functions monotone of arbitrarily high order in (a, b) form the class of analytic functions in (a, b) which can be analytically continued into the whole upper halfplane and map the latter into itself.

In the first part of the paper the authors use Hamburger's moment problem to give a new proof of the above described characterization of functions monotone of abritrarily high order. They further show that if instead of finite matrices X operators in Hilbert space are considered and $f(X)$ is defined in the usual fashion, monotonicity of the operator function $f(X)$ means the same as monotonicity of arbitrarily high finite order.

In the second part of the paper convex operator functions are considered. $f(X)$ is convex if

$$f[(1+t)A+tB]\leqq(1-t)f(A)+tf(B).$$

F. Kraus gave in his dissertation [see Math. Z. **41** (1936), 18–42] necessary and sufficient conditions for convexity of order n. Using his results the authors derive a very simple characterization of functions which are convex of arbitrarily high order. The difference quotient

$$(f(x)-f(x_0))/(x-x_0)$$

must be monotone of arbitrarily high order for some x_0 from (a, b) and this is also sufficient. Remark: The last two lines on p. 69 should read $\cdots$ necessarily the quadratic function

$$f(x)=f(x_0)+\frac{f'(x_0)}{1}(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2.$$

C. Loewner (Palo Alto, Calif.).

Korányi, A. On a theorem of Löwner and its connections with resolvents of selfadjoint transformations. Acta Sci. Math. Szeged **17** (1956), 63–70.

Löwner [Math. Z. **38** (1934), 177–216] proved the following theorem. The class of all monotone functions of self adjoint operators on a Hilbert space with spectra on $(-1, 1)$ is identical with the class C_P of all real valued continuously derivable functions on $(-1, 1)$ which are positive definite in the sense that if $x_1, x_2, \cdots, x_n$ are points of $(-1, 1)$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ are complex numbers then $\sum_{i,j=1}^n k(x_i, x_j)\alpha_i\bar{\alpha}_j\geqq0$, where $k(x, x)=f'(x)$ and $k(x, y)=(f(x)-f(y))/(x-y)$ for $x\neq y$. It is known also (Bendat and Sherman, Trans. Am. Math. Soc., **79** (1955) pp. 58–71) that the class C_P is identical with the class C_I of functions representable in the form

$$f(x)=f(0)+\int_{-1}^{1}x(1-tx)^{-1}d\mu(t)$$

with bounded non-decreasing $\mu(t)$. The first theorem of the present paper gives a new and elegant proof of the quivalence of C_P and C_I by using Hilbert space methods. By suitable alterations of the proof of the first theorem the author arrives at the proof of his second theorem which states that if T_x is a one parameter family of bounded symmetric operators defined on $(-1, 1)$ to a Hilbert space $\mathscr{H}$ then necessary and sufficient conditions that there exist a larger Hilbert space $\mathscr{H}^1\supseteq\mathscr{H}$ and a self adjoint operator A such that $T_x=P^1Q_x{}^1$ where $Q_x{}^1=x(I-xA)^{-1}$ and P^1 is the projection of $\mathscr{H}^1$ onto $\mathscr{H}$ are (a) T_x has a weakly continuous weak derivative T_x', (b) $T_0=0$, $T_0'=I$, (c) for any finite system of points $\{x_i\}\subset(-1, 1)$ and elements $\{f_i\}\in\mathscr{H}$, $i=1, 2, \cdots, n$; $\sum_{i,j=1}^n(K(x_i, x_j)f_i, f_j)\geqq0$ where $K(x, x)=T_x'$, $K(x, y)=(T_x-T_y)/(x-y)$ if $x\neq y$.

R. E. Fullerton.

Sz.-Nagy, Béla. Remarks to the preceding paper of A. Korányi. Acta Sci. Math. Szeged **17** (1956), 71–75.

The two results of the paper reviewed above are generalized by replacing the assumption of continuous differentiability of f and T_x by the respective hypotheses that f is continuous and derivable almost everywhere on $(-1, 1)$ and that T_x is weakly continuous and weakly derivable almost everywhere on $(-1, 1)$. The author shows that the conclusions of Korányi are still valid under these weaker hypotheses.

R. E. Fullerton.

Chong, F. Common eigenvectors of commuting operators. Austral. J. Sci. **19** (1956), 9–10.

A proof, simpler than previous ones, is given for the theorem that two commuting normal operators on a Hilbert space have a complete orthonormal set of eigenvectors in common, so that, in particular, two commuting normal matrices can be simultaneously diagonalized by the same unitary similarity transformation.

Kramer, Vernon A. Asymptotic inverse series. Proc. Amer. Math. Soc. **7** (1956), 429–437.

Let H_i $(i\geqq0)$ be positive operators in Hilbert space and let $H_0\geqq I$. Then the inverse of the operator

$$H(t)=H_0+tH_1+t^2H_2+\cdots\quad(0\leqq t\leqq t_0),$$

(it is assumed that the domain of $H(t_0)$ is dense), has a formal expansion $H(t)^{-1}=A_0+tA_1+t^2A_2+\cdots$. It is shown that if the vector ϕ is in the domains of the operators $A_1, \cdots, A_N$ and if, for each $i<N$, $A_i\phi$ is in the domain of $H(t)$ then,

$$\lim_{t\to0}\|\tilde{H}(t)^{-1}-\sum_{i=0}^{N}t^iA_i\phi\|=0,$$

where $\tilde{H}(t)$ is the Friedrichs extension of $H(t)$. Under further restrictions error estimates are given. A similar asymptotic expansion is obtained for the inner product $(\tilde{H}(t)^{-1}\phi, \psi)$. *N. Dunford* (New Haven, Conn.).

Senft, Walter. Über die Einführung des Kongruenzbegriffes in der Theorie der linearen Räume. Comment. Math. Helv. **30** (1956), 73–97.

Starting from a very general notion of congruence in vector spaces, the author discusses various specialisations of it by means of axioms. We state one of his main results. Let $\cong$ denote a congruence relation in a real vector space L, let us write $x \perp y$ if and only if $x+y \cong x-y$, and let the three following conditions be fullfilled: (I) $x \cong z$ and $y \cong z$, implies $x \cong y$; (II) $x \perp y$, implies $\lambda x \perp y$; (III) for all x and $u \neq 0$, there exists a unique normal projection of x on u, i.e. a unique real λ such that $x-\lambda u \perp u$. Then, there exists a definite bilinear form $\Phi(x, y)$ such that $x \cong y$ is equivalent with $\Phi(x, x) = \Phi(y, y)$; the converse is obviously true. Here, L can be either finite or infinite dimensional; it should be noted that the remainder of the paper deals mostly with 2-dimensional, not necessarily real spaces. *J. L. Tits.*

Ogasawara, Tôzirô; and Maeda, Shûichirô. A generalization of a theorem of Dye. J. Sci. Hiroshima Univ. Ser. A. **20** (1956), 1–4.

The author generalizes a theorem of H. A. Dye [Duke Math. J. **20** (1953), 55–69; MR **14**, 659] on the extension of a group isomorphism between the unitary groups of certain rings of operators in Hilbert space by showing that the finiteness assumption made by Dye on the rings can be dropped. *F. I. Mautner* (Baltimore, Md.).

Maeda, Shûichirô. Lengths of projections in rings of operators. J. Sci. Hiroshima Univ. Ser. A. **20** (1956), 5–11.

The author treats under more general circumstances the notion of length of a projection in a ring of operators due to Dye [Trans. Amer. Math. Soc. **72** (1952), 243–280; MR **13**, 662]. Using a definition due to Ogasawara [J. Sci. Hiroshima Univ. Ser. A. **19** (1955), 255–272; MR **18**, 137], he obtains a decomposition for an arbitrary ring as a direct sum of rings of uniform length, a result established earlier for type III rings by Griffin [Trans. Amer. Math. Soc. **79** (1955), 389–400; MR **17**, 66]. He then defines a length -function for a ring and obtains from it a different definition of the coupling invariant used by Griffin and Pallu de la Barriere, in treating multiplicity theory for type II rings. *I. E. Segal* (Chicago, Ill.).

See also: Roelcke, p. 571; Feller, p. 575; Temple, p. 580; Friedman, p. 584; Foias, p. 586.

TOPOLOGY

General Topology

Jaworowski, J. W. On antipodal sets on the sphere and on continuous involutions. Fund. Math. **43** (1956), 241–254.

The paper is a detailed presentation of results previously announced [Bull. Acad. Polon. Sci. Cl. III. **3** (1955), 247–250, 289–292; MR **17**, 289, 653]. The author first introduces the notion of an antipodal system and then proves several theorems on subsets of the n-sphere S^n and a special case of a fixed point theorem by P. A. Smith. (The concept of an antipodal system is essentially contained in the special homology theory of P. A. Smith. In fact, a subset of S^n carries an antipodal k-system if and only if it contains an antipodal compact set X such that the natural homomorphism of $\bar{H}_k(X)$ into $\bar{H}_k(S^n)$ is not trivial, where $\bar{H}_k$ is the kth special homology group with respect to the antipodal map and integers mod 2 are used as coefficients.) Let A, B be subsets of S^n carrying an antipodal p-system and an antipodal q-system respectively. The main theorems are: 1) If $p+q=n-1$ and $A \cap B = \emptyset$ then A and B are linked. 2) If $p+q \geqq n$, then $A \cap B \neq \emptyset$. {The question raised by the author about whether $A \cap B$ carries an antipodal $(p+q-n)$-system for $p+q \geqq n$ has a positive answer as seen in a paper by the reviewer [Ann. of Math. (2) **62** (1955), 271–283; MR **17**, 289]. From this result both 2) and 1) can be shown without much difficulty.} The theorems related to the Borsuk's theorems on antipodal points are not different from those obtained by the reviewer [ibid. **60** (1954), 262–282; MR **16**, 502]. *C. T. Yang* (Philadelphia, Pa.).

Kapuano, Isaac. Propriétés des ensembles toujours de première catégorie. C. R. Acad. Sci. Paris **244** (1957), 31–34.

Famille non stationnaire décroissante d'ensembles-F_σ. Tout ensemble toujours de première catégorie a une puissance $\leqq \aleph_1$. L'hypothèse du continu est équivalente à la proposition suivante: Il existe un ensemble E dépourvu d'ensemble parfait, ayant la puissance du continu et réunion de $\aleph_1$ ensembles jouissant de la propriété de Baire au sens restreint. Rectification d'un énoncé précédent [mêmes C. R. **242** (1956), 1833–1836, lemme 1; MR **17**, 1065]. (Résumé de l'auteur.) *L. Gillman.*

Kurata, Y.; and Kato, M. A note on covering spaces. J. Fac. Sci. Hokkaido Univ. Ser. I. **13** (1956), 65–67.

In a remark following the proof of Proposition I, Chevalley [Theory of Lie groups, v. I, Princeton, 1946, p. 51; MR **7**, 412] announced that this proposition can be deduced from his principle of monodromy (ibid. p. 46) in the special case where the space $\mathfrak{B}$ is normal. The present authors propose a modified principle of monodromy from which the proposition in question is deduced without assumption of normality, their principle is as follows: Let $\mathfrak{B}$ be a simply connected space. Assume that we have assigned to every $p \in \mathfrak{B}$ a non-empty set E_p. Let $\mathfrak{U}$ be an open basis of $\mathfrak{B}$ consisting of connected sets. Assume furthermore that we have assigned to every triple $(U; p, q)$, where $U \in \mathfrak{U}$ and $p, q \in U$, mapping $\varphi_{p,q}^{(U)}$ of E_p into E_q in such a way that the following conditions are satisfied: 1) If $U \in \mathfrak{U}$ and $p, q, r \in U$, we have $\varphi_{p,r}^{(U)} = \varphi_{q,r}^{(U)} \cdot \varphi_{p,q}^{(U)}$; 2) if U, $V \in \mathfrak{U}$ and $U \supset V$, we have $\varphi_{p,q}^{(U)} = \varphi_{p,q}^{(V)}$ for all p, $q \in V$; 3) each $\varphi_{p,p}^{(U)}$ is the identity mapping. Then there exists a mapping ψ which associates with every $p \in \mathfrak{B}$ an element $\psi(p) \in E_p$ in such a way that $\psi(q) = \varphi_{p,q}^{(U)}(\psi(p))$ where $U \in \mathfrak{U}$ and $p, q \in U$. Moreover, if $p_0 \in \mathfrak{B}$ and $e_{p_0}{}^0 \in E_{p_0}$ are given, then the mapping ψ is uniquely selected in such a way that $\psi(p_0) = e_{p_0}{}^0$.

W. W. S. Claytor (Washington, D.C.).

Tominaga, Akira. On some properties of non-compact Peano spaces. J. Sci. Hiroshima Univ. Ser. A. **19** (1956), 457–467.

A Peano space is a space R which is locally compact, locally connected, separable and metric. If R is also compact, then R is a Peano continuum. Given a space S, and a point p at which S is locally connected, the degree $D(S, p)$ of S at p is defined by a limiting process which generalizes the usual definition of the order of a linear graph at a point. If R is a Peano space, then R has a one-point compactification $R^*=R\cup p$, and R^* is locally connected at p. If R is not compact, then the degree $D(R)$ of R is defined as $D(R^*, p)$; if R is compact, then $D(R)$ is defined to be zero. The author's main results are as follows. Hereafter, R always denotes a Peano space.

(1) R is non-compact if and only if for each point x of R there is a halfopen arc A in R, having x as its end-point, such that A forms a closed set in R. (2) If the metric for R is convex, and $D(R)\geqq 2$, then R contains a straight line. (Here a "straight line" is not necessarily unbounded relative to the given metric.) A metric space has property S if for each $\varepsilon>0$ S can be expressed as the union of a finite collection of connected sets, each of which is of diameter $<\varepsilon$. (3) Every R has a bounded convex metric relative to which R has property S. (4) If R is simply connected, then so also is the Freudenthal compactification of R. [For the definition of the Freudenthal compactification (which assigns "endpoints" to R) see Math. Z. **33** (1931), 692–713.] (5) Let M be a closed set in R, and let σ be a convex metric for M. Then σ can be extended to give a convex metric for R. (3) and (5) generalize results of Bing [Bull. Amer. Math. Soc. **55** (1949), 1101–1110; Duke Math. J. **14** (1947), 511–519; MR **11**, 733; **9**, 521]. {The author mistakenly credits the reviewer with having given a proof of the convexification theorem for Peano continua, in Bull. Amer. Math. Soc. **55** (1949), 1111–1121 [MR **11**, 734]; in fact, the argument there given for the grille-decomposition theorem (on which the rest of the paper depended) was erroneous [see the acknowledgment in Proc. Amer. Math. Soc. **2** (1951), 838; MR **13**, 265].} *E. Moise.*

Mrówka, S. On quasi-compact spaces. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 483–484.

Let X be a completely regular space. If every continuous real function on X is bounded, then X is quasi-compact. Suppose it is true that if X is imbedded in a completely regular space $X\cup\alpha$, where α is a point at which $X\cup\alpha$ satisfies the first axiom of countability, then α is isolated in $X\cup\alpha$. Then we say that X is countably absolutely closed [this generalizes a definition of "absolutely closed", due to Alexandroff and Urysohn, Verh. Akad. Wetensch Amsterdam. Afd. Natuurk. Sect. 1. **14** (1929), no. 1].

Theorem. If X is completely regular, then X is quasi-compact if and only if X is countably absolutely closed.

Theorem. If X is normal, then X is compact if and only if X is countably absolutely closed. *E. Moise.*

Anderson, R. D. Atomic decompositions of continua. Duke Math. J. **23** (1956), 507–514.

Let M be a continuum, and let G be a continuous decomposition of M into continua. Suppose that for each subcontinuum K of M, and each element g of G, either $g\subset K$ or $K\subset g$. Then G is said to be atomic.

Theorem. Let S be a continuum imbedded in a space R, such that R is either a Hilbert cube or a triangulable Euclidean manifold. Let K be either a positive integer or ∞; and suppose that $K<\dim(R)$ if $\dim(R)<\infty$. Let ε be a positive number. Then there exists a continuum M in R, and an open mapping θ, of M onto S, such that (1) the collection E of inverse images $\theta^{-1}(p)$ for $p\in S$ is a continuous atomic decomposition of M into continua, (2) each element of E contains a K-cell, (3) $\dim(M)=K$, and (4) for $p\in S$, the Hausdorff distance between p and $\theta^{-1}(p)$ is less than ε. *E. Moise* (Princeton, N.J.).

Ryll-Nardzewski, C. A remark on the mixing theorem. Bull. Acad. Polon. Sci. Cl. III, **3** (1955), 297–298.

Let T be an indecomposable (metrically transitive) measure preserving transformation in a finite positive measure space (X, B, m). Each of the following two conditions, which are known to be equivalent to the weak mixing property of T if T is one-to-one, are here shown to be equivalent without this assumption: (a) The square of T, i.e. $(T\times T)(x, y)=(Tx, Ty)$, is indecomposable with respect to the direct product measure $m\times m$; (b) for any complex number λ, the function $f(x)=$const is the only one for which $f(Tx)=\lambda f(x)$. *N. Dunford.*

See also: Banaschewski, p. 551; Popruženko, p. 551; Eyraud, p. 551; Sikorski and Traczyk, p. 555; Artin, p. 564; Stoïlow, p. 568; Rudin, p. 587.

Algebraic Topology

Inoue, Yoshiro. On a condition of the extendability of a cross section. Math. Japon. **4** (1956), 1–4.

Let $p: E\to B$ be a fibre space in the sense of J.-P. Serre, where B is an arcwise connected CW-complex. Assume a cross section ψ is defined over the $(m-1)$-skeleton of B. The author gives some necessary and sufficient conditions for the extendability of ψ to a cross section over the m-skeleton of B. *W. S. Massey.*

Papakyriakopoulos, C. D. On Dehn's lemma and the asphericity of knots. Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 169–172.

In 1910 Max Dehn published what purported to be a proof of the proposition that if D is a combinatorial singular 2-cell in a triangulated 3-manifold M, and D has no singularities on its boundary, then the boundary of D is unknotted [Math. Ann. **69** (1910), 137–168, *p.* 147]. This was known as the Dehn lemma. It can be stated more precisely as follows:

Lemma. Let σ^2 be a 2-simplex with boundary β. Let f be a piecewise linear mapping of σ^2 into the triangulated 3-manifold M. Suppose that there is a neighborhood U of β in the space σ^2, such that (1) $f|U$ is a homeomorphism and (2) $f(U)\cap f(\sigma^2-U)=0$. Then the polygon $f(\beta)$ is the boundary of a polyhedral disk in M.

Nineteen years after the appearance of Dehn's paper it was found by Kneser [Jber. Deutsch. Math. Verein. **38** (1929), Abt. 1, 248–260, p. 260] that Dehn's proof of the lemma was erroneous; and since then the lemma has represented a celebrated problem. In the paper under review, the author announces an affirmative solution of this problem, and gives indications of a proof which will later be published in full. The proof makes use of a result due to Johansson [Math. Ann. **115** (1938), 658–669] to the effect that the lemma reduces to the case in which M is orientable.

As Dehn correctly proved, it is a consequence of the lemma that the knot-type of the circle in the 3-sphere is characterized by the knot-group. That is to say, if K is a (polygonal) knot in S^3, and the fundamental group $\pi_1(S^3-K)$ is infinite cyclic, then K is the boundary of a polyhedral disk.

Let F be a non-empty, closed, proper subset of S^3. If there is a 2-sphere S^2 in S^3-F such that each of the components of S^3-S^2 intersects F, then we say that F is geometrically splittable. If $\pi_1(S^3-F)$ is the free product of two non-trivial groups, then we say that F is algebraically splittable. If U is an open subset of S^3, and the second homotopy group $\pi_2(U)$ is$=0$, then we say that U is aspherical.

Theorem. Let U be a non-empty, open, proper connected subset of S^3. Then U is aspherical if and only if S^3-U is not geometrically splittable.

Corollary. If F is a non-empty, connected, closed proper subset of S^3, then every component of S^3-F is aspherical.

This is the solution of a problem of Eilenberg [Fund. Math. **28** (1936), 233–242, p. 241].

Corollary. If F is a connected graph, or a knot, then S^3-F is aspherical.

By a link we mean a finite union of disjoint knots. The following result is the solution of a problem of Higman [Quart. J. Math. Oxford Ser. **19** (1948), 117–122; MR **9**, 606].

Theorem. Let K be a link in S^3. The following three statements are equivalent: (i) S^3-K is not aspherical. (ii) K is geometrically splittable. (iii) K is algebraically splittable.

The proofs of the above results are based in part on the following theorem. The Sphere Theorem. Let M be an orientable triangulated 3-manifold with boundary. (The boundary of M may be empty, and M is not necessarily compact.) Suppose that $\pi_2(M)\neq 0$; and suppose that M is combinatorially imbeddable in a 3-manifold N which has the property that the first homology group of every non-trivial subgroup of $\pi_1(N)$ has an element of infinite order. Then there is a polyhedral 2-sphere S in M such that S is not homotopic to zero in M.

The following result verifies a conjecture of Hopf. Theorem. If U is an open connected subset of S^3, then $\pi_1(U)$ has no element of finite order.

Finally, it is shown that if K is a knot in S^3, then $\pi_1(S^3-K)$ has either one end or two ends, according as K is knotted or unknotted. *E. Moise* (Princeton, N.J.).

Cockcroft, W. H. The cohomology groups of a fibre space with fibre a space of type $\mathcal{K}(\pi, n)$. Proc. Amer. Math. Soc. **7** (1956), 1120–1126.

Let X be a topological space with two non-vanishing homotopy groups, say π_m, π_n in dimensions m, n respectively ($m>n>1$). The author calculates the cohomology groups of X with coefficients in a field up to dimension $m+2$. In effect only the three dimensions m, $m+1$, $m+2$ are significant, and the author determines the corresponding cohomology groups explicitly up to group extensions. Since X has the same homotopy type as a fibre space with base of type $\mathcal{K}(\pi_n, n)$ and fibre of type $\mathcal{K}(\pi_m, m)$, it is sufficient to consider the case of a general fibre space with fibre of type $\mathcal{K}(\pi_m, m)$. Using standard spectral sequence arguments the author determines the cohomology of such a fibre space up to dimension $m+2$.

M. F. Atiyah (Cambridge, England).

See also: Karpelevič, p. 583.

GEOMETRY

Geometries, Euclidean and other

★**Strickler, Walter. Über die endlichen klein-desarguesschen Zahlsysteme.** Dissertationsdruckerei Leemann AG, Zürich, 1955. 33 pp.

A "small-Desargues" system Z is a set of elements, called numbers, which satisfy those laws of addition and multiplication which can be obtained through Hilbert's construction from the "small-Desargues theorem", i.e. the special case where both corresponding sides and the lines connecting corresponding vertices are parallel. Finite Z are characterized by the following properties: Z is an abelian group under addition. M_1: $ab=ab'$ or $ca=c'a$ and $a\neq 0$ imply $b=b'$ or $c=c'$. M_2: There is a left and a right unit. D: $a(b+c)=ab+ac$ (but not the right distributive law). It follows that the right and left units coincide and that division is possible. M_1 and M_2 are independent of the remaining axioms. Three subsystems (satisfying all the axioms) are discussed. A: Consists of those $a\in Z$ for which $(xy)a=x(ya)$ for all x, $y\in Z$. Δ: The intersection of A and those elements d for which $(x+y)d=xd+yd$ for all $x, y\in Z$. Finally, Π is the smallest subsystem of Z. It is shown that Z is a right vector space over Δ. If, and only if, all elements of Z commute with all elements of Π, Z is also a left vector space over Δ. Special systems Z, in particular those of Veblen, MacLagen-Wedderburn, are discussed. A Z with distinct A, Δ, Π has at least 256 elements. Z is proper if the right distribution law does not generally hold. There are no proper commutative or anti-commutative Z. *H. Busemann* (Los Angeles, Calif.).

Neculcea, M. Extensions de l'axiome de congruence des triangles dans le plan. Rev. Math. Phys. **2** (1954), 133–137 (1955).

In Hilbert's axioms for euclidean plane geometry, the congruence axiom III.5 [Grundlagen der Geometrie, 7. Aufl., Teubner, Leipzig, 1930, p. 14] can be "reduced" by making the assumption that all triangles under consideration differ from a given fixed triangle or, more generally, do not belong to a given set, such that any bounded region of the plane contains only a finite number of triangles of the set. On the other hand, an explicit model shows that one cannot exclude all the triangles having a given point as one vertex. *J. L. Tits* (Brussels).

Freudenthal, Hans. Neuere Fassungen des Riemann-Helmholtz-Lieschen Raumproblems. Math. Z. **63** (1956), 374–405.

Let R be a connected locally compact topological space and let F be a transitive group of homeomorphisms of R. The reviewer has determined [Bull. Soc. Math. Belg. **5** (1952), 44–52; **6** (1953), 126–127; Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 8° **29** (1955), no. 3; MR **15**, 334; **16**, 501; **17**, 874] all pairs (R, F) satisfying the following conditions: (S) For any two closed subsets of R, A and B, with $A\cap B=\emptyset$, there exists an open subset U of R such that for no $f\in F$, $f(U)\cap A\neq\emptyset$ and $f(U)\cap B\neq\emptyset$ simultane-

ously [this is the author's interesting formulation for a condition due to Kolmogorov, Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1930**, 208–210]; (V) F is complete (for a uniform structure invariant under F which can be introduced on R as a consequence of (S)); (Z^*) let J be the group of all elements of F which leave fixed a given point x_0; then, of any two orbits of J, the one separates the other from x_0. The author generalizes this result by substituting in (Z^*) the weaker condition: (Z) There exists an orbit of J which separates R. This substitution does not give rise to new pairs (R, F), so that the list of all (R, F) satisfying (S), (V) and (Z) is the one given in MR **15**, 334; as a consequence of this, if R denotes now a connected, locally compact metric space and if there exists a number $\gamma>0$, smaller than the diameter of R, such that R is homogeneous with respect to the pairs of points at distance γ, then, R is two-point homogeneous in the sense of Birkhoff-Wang, i.e. it is a euclidean space, a sphere, or a real, complex, quaternion or octave elliptic or hyperbolic space [cf. loc. cit.]. The author's proof is shorter than the reviewer's; it should be noted, however, that the latter makes a wider use of elementary means. The paper starts with a historical introduction on the Riemann-Helmholtz-Lie problem. *J. L. Tits.*

★**Yaglom, I. M. Geometričeskie preobrazovaniya. II. Lineĭnye i krugovye preobrazovaniya. [Geometrical transformations. II. Linear and circular transformations.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956. 612 pp. 10.45 rubles.

This second volume, like the first one, is intended for superior high school students and high school teachers, and the material is strictly adapted to the background of Russian high schools. The methods are purely synthetic and much familiarity with the geometry of the circle is assumed, but other conic sections are not considered. The first part deals with affine and projective transformations of E^2 and E^3. Projective spaces are never introduced axiomatically, the euclidean space is completed by elements at infinity. The second part deals with the elementary aspects of Laguerre geometry, in particular, reflections and inversions in circles are studied in detail. Hyperbolic geometry is treated twice, first its Klein model in part 1, and then its Poincaré model in part 2. There are about 200 problems, some of them very interesting, formulated in the text proper (pp. 1–354) and their solutions, often with digressions into related subjects, occupy the remaining space (pp. 355–605).
H. Busemann (Los Angeles, Calif.).

Fabricius-Bjerre, Fr. Some theorems of J. Hjelmslev on plane, skew, and spherical quadrilaterals. Nordisk Mat. Tidskr. **4** (1956), 139–148, 176. (Danish. English summary)

Let $Q_i=A_iB_iC_iD_i$ ($i=1$, 2) be two (not necessarily convex and possibly self-intersecting) quadrangles in E^2. Put $A_iB_i=a_i$, $B_iC_i=b_i$, $C_iD_i=c_i$, $D_iA_i=d_i$, $A_iC_i=e_i$, $B_iD_i=f_i$. Let

$$a_1+a_2=b_1+b_2=c_1+c_2=d_1+d_2,$$
$$a_1+b_1+c_1+d_1=a_2+b_2+c_2+d_2,\ e_1=e_2,$$

and assume (*) that either both Q_i are decomposed, or both not decomposed by the diagonal A_iC_i. Then 1) $f_1=f_2$, 2) the distances of the centers of the diagonals are equal, 3) the products of the areas of two triangles with common corresponding diagonals are equal. These facts are true for quadrangles in E^3 provided (*) is replaced by the condition that the angles between the planes $A_iB_iC_i$ and $A_iD_iC_i$, $i=1$, 2, are equal. Moreover, the tetrahedra $A_iB_iC_iD_i$ have equal volumes. The results on E^2 are extended to S^2. *H. Busemann* (Los Angeles, Calif.).

Zacharias, Max. Die Aufgabe von Senkatachalam Jyer. Bull. Soc. Roy. Sci. Liège **25** (1956), 605–606.

A solution with ruler and compasses is given for a problem in plane geometry originating with Senkatachalam Jyer and discussed analytically by H. Lorent (same Bull. **24** (1955) 14–24; listed in MR **16**, 1044).

Marmion, A. Sur le "module" d'un tétraèdre. Mathesis **65** (1956), 519–526.

The "module" of a tetrahedron is a real number $M\geqq-1$, defined in terms of the sides and angles in such a way that its value determines certain relations among the areas of the faces; e.g., if $M>1$, then the sum of the greatest and least areas is less than the sum of the other two.

Bilo, J. Sur l'affinité orthologique. Mathesis **65** (1956), 509–516.

Lorent, H. Sur les tangentes et les normales à deux coniques conjuguées. Bull. Soc. Roy. Sci. Liège **25** (1956), 595–604.

Cavallaro, Vincenzo G. Sui segmenti Torricelliani. Giorn. Mat. Battaglini (5) **4(84)** (1956), 81–91.

Let BCL, CBM, ABN (BCL', CAM', ABN') be the three equilateral triangles constructed on the sides of a triangle ABC and external (internal) to ABC. We have:

$$AL=BM=CN(=T),\ AL'=BM'=CN'(=T').$$

During the last three decades the author has published a series of papers, overlapping considerably in contents, in which he has been defending the contention that the segments T, T', which he calls the Torricelli segments, play an important role in the geometry of the triangle. In support of this claim he has derived a considerable number of relations involving T and T'. We shall quote some of them, more or less at random, from the paper under review (in a footnote the author lists most of his previous articles on this subject).

If a, b, c are the sides of the triangle ABC, we have

$$a^2+b^2+c^2=T^2+T'^2.$$

Denoting by 2φ, 2ψ the axes, and by F, F' the foci of the incribed Steiner ellipse of ABC, we have

$$2\varphi=(T+T')/3,\ 2\psi=(T-T')/3,\ FF'=2(TT')^{\frac{1}{2}}/3.$$

Sin $\omega=(T^2-T'^2)/2(T^4+T'^4+T^2T'^2)^{\frac{1}{2}}$, where ω is the Brocard angle of ABC.

If T, T' and T_1, T_1' are the pairs of Torricelli segments of two triangles ABC and $A_1B_1C_1$, the necessary and sufficient condition that a) the two triangles shall have equal Brocard angle is

$$T:T'=T_1:T_1',$$

and b) the Brocard angle of $A_1B_1C_1$ shall be equal to the first Steiner angle of ABC is

$$T:T'=T_1^2:T_1'^2.$$

If a right trianglar prism has ABC for its base and is cut by a plane so that the section is an equilateral triangle, we have, denoting by L the side of the equilateral triangle

and by θ the angle between the planes of the two triangles:

$$L=3^{\frac{1}{2}}(T+T')/3, \quad \cos\theta=(T-T')/(T+T').$$

Both these relations were given by the author before [Math. Gaz. **36** (1952), 273–275, p. 274; Mathesis **62** (1953), 21–30, p. 29; MR **14**, 785].

The centers of the two triads of equilateral triangles considered before form two equilateral triangles which the author calls the triangles of Napoleon (Bonaparte). If l, l' are the sides of those two triangles, we have:

$$L=l+l'=2\varphi\cdot 3^{\frac{1}{2}}.$$

These two "bellissima proprietà" were given by the author before [Mathesis **62** (1953), p. 29]. *N. A. Court.*

Cundy, H. Martyn. Unitary construction of certain polyhedra. Math. Gaz. **40** (1956), 280–282.

Cutting a solid cuboctahedron into two halves by the plane of one of its four "equatorial" hexagons, the author obtains a pair of pieces of a shape that can usefully be cemented together like bricks in a wall. Eight such pieces, joined along their square faces, make a truncated octahedron with its square faces missing, revealing the vertices of an octahedron inside. In this manner he exhibits an interesting connection between the two honeycombs $t_1\delta_4$ and $t_{1,2}\delta_4$ [Coxeter, Math. Z. **46** (1940), 380–407, pp. 403–404; MR **2**, 10]. The former consists of octahedra and cuboctahedra; the latter, of truncated octahedra alone.

Slightly more complicated bricks of the same general character are used to set up other models which neatly illustrate Mrs. Stott's construction for reflexible Archimedean solids [W. W. R. Ball, Mathematical recreations and essays, 11th ed. Macmillan, London, 1939].

H. S. M. Coxeter (Toronto, Ont.).

Gleason, Andrew M. Finite Fano planes. Amer. J. Math. **78** (1956), 797–807.

Let $\mathcal{P}$ be a projective plane. The plane is called a Fano plane provided the diagonal points of every quadrangle are collinear. The following has been conjectured for some time: Every finite Fano plane is Desarguesian. The main purpose of this paper is to prove this conjecture. The technique consists in showing that the existence in a finite plane of certain collineations of a special type establishes the small, and hence the full, theorem of Desargues, and that finite Fano planes have these required collineations.

The introduction is devoted to a brief historical summary of the theory of projective planes, and indicates the gap filled by the derivation of this conjecture. In Section 1 the collineation group is studied. For a given point x and line L of a projective plane $\mathcal{P}$, G_{xL} is defined as the group of all collineations which leave x fixed line-wise and L fixed point-wise. A series of lemmas is established, culminating in Theorem 1.8. "Let $\mathcal{P}$ be a finite projective plane. Suppose, for every point x and line L with $x \in L$, that G_{xL} is non-trivial. Then $\mathcal{P}$ is Desarguesian." Section 2 concerns projections and studies the Reidemeister configuration, referred to as configuration G. The main result here is Theorem 2.5. "If configuration G holds in $\mathcal{P}$ and the order of $\mathcal{P}$ is a prime power, then $\mathcal{P}$ is Desarguesian." Section 3 introduces the Fano plane, and the main result then follows from the previous considerations without great difficulty. *H. J. Ryser.*

Semyanistyĭ, V. I. Parabolic congruences of straight lines. Trudy Sem. Vektor. Tenzor. Anal. **10** (1956), 259–268. (Russian)

Les congruences paraboliques dont il est question sont les congruences de droites définies dans l'espace projectif réel P_{2n+1} de dimension impaire par deux variétés linéaires à n dimensions infiniment voisines (le langage des variétés infiniment voisines n'est pas utilisé par l'auteur); l'auteur montre que les droites d'une telle congruence peuvent être représentées par les points d'un espace projectif à n dimensions $P_n(\varepsilon)$ sur les nombres duaux $a+b\varepsilon$ ($\varepsilon^2=0$). Etant donnée dans l'espace projectif P_{4n+3} une hyperquadrique non dégénérée S_{4n+3}, d'indice maximum, qui sera considérée comme l'absolu d'un espace non-euclidean, il existe des „congruence paraboliques paratactiques", dont les deux variétés linéaires de base infinement voisines appartiennent à S_{4n+3} (ce qui signifie que les droites de la congruence sont toutes tangentes à S_{4n+3}); l'auteur montre que les couples de droites d'une telle congruence ont un invariant métrique (nombre réel attaché à ces couples et invariant pour les projectivités de P_{4n+3} conservant simultanément S_{4n+3} et la congruence donnée), auquel correspond dans $P_{2n+1}(\varepsilon)$ un invariant de couples de points qui n'est autre que la distance pour une métrique unitaire définie dans $P_{2n+1}(\varepsilon)$. *J. L. Tits* (Bruxelles).

Bompiani, Enrico. Complessi lineari e fasci di complessi lineari di rette in S_n. Rend. Mat. e Appl. (5) **15** (1956), 1–23.

Il presente lavoro riproduce una parte di un corso di lezioni tenuto dall'A. nell'Università di Roma. Oltre a risultati già noti relativi agli spazi totali ed allo spazio singolare di un complesso lineare di rette di S_n, riottenuti con metodo conciso ed elegante mercè l'introduzione dell'operazione di „accoppiamento" (operazione che riguarda una s-pla non ordinata di rette di S_n e che consiste nel costruire una certa forma s-lineare nelle coordinate di tali rette), l'Autore arreca nuovi contributi allo studio dei fasci di complessi lineari di rette di S_{2r-1}. Precisamente: a) dimostra che un generico fascio di complessi lineari di rette di S_{2r-1} è determinato dalle sue r rette singolari (distinte ed appartenenti ad S_{2r-1}), e cioè dalle rette singolari degli r suoi complessi singolari, e dallo spazio polare di un punto P (S_{2r-3} sostegno degli ∞^1 S_{2r-2} polari di P rispetto ai complessi del fascio stesso); b) assegna un significato geometrico agli $r-3$ invarianti del fascio ed una costruzione geometrica del fascio stesso.

Relativamente ai fasci di complessi lineari tutti singolari ma di prima specie, l'Autore dimostra che le ∞^1 rette singolari dei complessi di un tal fascio, che sono congiunte da uno spazio totale del fascio (cioè di tutti i suoi complessi), appartengono al più ad un S_r; ciò è immediata conseguenza di un lemma affermante che un complesso lineare di rette di S_{2r-1} avente un S_{r+k} totale ha necessariamente un S_{2k+1} singolare. Le rette singolari del fascio ricoprono, se $r \geq 3$, una rigata d'ordine $r-1$ (di S_r).

Per ciò che concerne i fasci speciali di complessi lineari di rette (fasci con rette singolari non tutte distinte) l'A. si limita a pochi cenni, indicando come il loro studio si possa ricondurre a quello di fasci generali, dopo aver precisato la nozione di rette singolari infinitamente vicine o coincidenti: e cioè una retta, un S_3 per essa ed una proiettività tra la punteggiata sulla retta ed il fascio di piani passanti per essa e contenuti nell'S_3.

D. Gallarati (Genova).

Bottema, O. Inequalities in the geometries of spheres and of lines. Nederl. Akad. Wetensch. Proc. Ser. A. **59** = Indag. Math. **18** (1956), 523–531.

The paper derives numerous determinantal inequalities concerning finite systems of spheres (perhaps degenerate) from an elementary theorem concerning the intersection of a quadric in $(n-1)$-dimensional projective space with a linear subspace. The well-known inequalities satisfied by the mutual powers of $N+2$ spheres, and the mutual distances of $N+2$ points in euclidean N-space follow as special cases. {Reviewer's note. The elements $(A_{N+2}A_{N+2})$, $(A_{N+1}A_{N+1})$ of the determinants in inequalities (14), (16), respectively, should be replaced by 0.}

L. M. Blumenthal (Columbia, Mo.).

Gheorghiu, O. Em. Au sujet d'un espace quasi euclidéen. Acad. R. P. Romîne. Baza Cerc. Şti. Timişoara. Stud. Cerc. Şti. Ser. I. **2** (1955), 27–36. (Romanian. Russian and French summaries)

L'auteur considère un système de nombres hypercomplexes $\omega=x+yM+zM^2$, $M=\|a_{ij}\|$ $(i, j=1, 2, 3)$. Les trois composants de ω sont $\omega_i=x+y\lambda_i+z\lambda_i^2$, où λ_i est une racine de l'équation caractéristique de M. Le module ρ de ω est défini par $\rho^3=\omega_1\omega_2\omega_3$. Introduction des fonctions exponentielles et de trois fonctions à deux arguments qui jouent le rôle des cosinus (un cas particulier sont les fonctions de Appell). Application à la géométrie de Humbert dont l'élément lineaire est défini par $ds^3=d\omega_1 d\omega_2 d\omega_3$, où $d\omega_i=dx+\lambda_i dy+\lambda_i^2 dz$; il existe trois plans isotropes.

O. Bottema (Delft).

Gheorghiu, O. Em. Détermination des objets géométriques linéaires de 2e classe en X_n. Acad. R. P. Romîne. Baza Cerc. Şti. Timişoara. Stud. Cerc. Şti. Ser. I. **2** (1955), 37–40. (Romanian. Russian and French summaries)

L'auteur détermine les fonctions les plus générales satisfaisantes une loi linéaire donnée pour les objets géométriques de deuxième classe $\Omega_{j_1\cdots j_s}^{i_1\cdots i_t}(x^i)$. A la condition $t=s+1$ il y a deux types de solution.

O. Bottema (Delft).

Myers, Wm. M., Jr. Functionals associated with a continuous transformation. Pacific J. Math. **6** (1956), 517–528.

Let T be a continuous transformation from a simply connected polygonal region R in the Euclidean plane into Euclidean three space, and denote by T_1, T_2, T_3 the three plane transformations obtained by following T with a projection onto the coordinate planes π_1, π_2, π_3, respectively. For each point z_i in π_i and each subset P of R which is either a Jordan region or a domain let $k(z_i, T_i, P)$ denote the number of essential maximal model continua for the point z_i under the transformation T_i which are contained in the interior of P. Denote by $g(T_i, P)$ the Lebesgue integral of $k(z_i, T_i, P)$ taken over the plane π_i, and put $G(T, P)$ equal to the square root of the sum of the squares of the terms $g(T_i, P)$. If Φ is a collection of subsets P of R each of which is either a Jordan region or a domain let $G(T, \Phi)$ denote the sum of the terms $G(T, P)$ taken over the collection Φ. Define $a_1(T)$, $a_2(T)$, $a_3(T)$, $a_4(T)$, $a_5(T)$, $a_6(T)$ to be the least upper bound of $G(T, \Phi)$ taken with respect to collections Φ which consist of (1) pairwise disjoint simply connected polygonal regions, (2) pairwise disjoint polygonal regions, (3) simply connected Jordan regions with pairwise disjoint interiors, (4) Jordan regions with pairwise disjoint interiors, (5) pairwise disjoint simply connected domains, (6) pairwise disjoint domains, respectively. The transformation T represents a continuous surface S, and for fixed j the value of $a_j(T)$ is constant for every representation T of S. Thus any one of the functionals $a_j(T)$ may be used as a value for the lower area of the surface S, and indeed $a_3(T)$ and $a_6(T)$ have been so used [Reichelderfer, Trans. Amer. Math. Soc. **53** (1943), 251–291; MR **4**, 213; T. Radó, Length and area, Amer. Math. Soc. Colloq. Publ., v. 30, New York, 1948; MR **9**, 505]. In this paper the author shows that the functionals $a_1(T)$, $a_2(T)$, $a_3(T)$, $a_4(T)$, $a_5(T)$, $a_6(T)$ all yield the same value.

P. V. Reichelderfer (Columbus, Ohio).

★ **Зеленин, Е.В. [Zelenin, E. V.] Начертательная геометрия и черчение. [Descriptive geometry and drawing.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1953. 524 pp. 16.35 rubles.

See also: Artin, p. 553; Fischer, p. 560; Hall, p. 560; Senft, p. 589; Egloff, p. 595; Graeub und Nevanlinna, p. 597; Ishehara, p. 599; Obata, p. 599; Noskov, p. 610; Shibata, p. 627.

Convex Domains, Integral Geometry

Santaló, L. A. On the chords of a convex curve. Rev. Un. Mat. Argentina **17** (1955), 217–222 (1956). (Spanish)

Let D be a closed plane convex curve bounding an area F and having a continuous radius of curvature with a positive minimum r_0. Consider all pairs of tangents of D forming the angle θ (containing D). The chord c connecting the points of contact of these tangents satisfies the inequality $c\geqq 2r_0\cos(\frac{1}{2}\theta)$ and particular values of c satisfy $c\geqq(F/2\pi)^{\frac{1}{2}}(1+\sin\theta+\cos\theta)$. The special case $\theta=\frac{1}{2}\pi$ was treated previously by J. W. Green [Portugal. Math. **10** (1951), 121–123; **11** (1952), 51–55; MR **13**, 577; **14**, 495]. The latter result is generalized to convex surfaces in E^3: particular values of the chord connecting the points of contact of two supporting planes forming the angle θ satisfy the inequality

$$c=(2^{5/2}\pi)^{-1}(1+\sin\theta+\cos\theta)M,$$

where M is the integral over the mean curvature.

H. Busemann (Los Angeles, Calif.).

Bredon, Glen E. The isoperimetric problem in the plane. Math. Mag. **30** (1956), 63–69.

An expository article.

Pleijel, Arne. On the division of convex figures. Nordisk Mat. Tidskr. **4** (1956), 149–151, 176. (Swedish. English summary)

Let the line (plane) L intersect the convex body K in E^2 (E^3) in an interior point. If m is the ratio of the distances of L from the supporting line (plane) of K parallel to L and k and g are the corresponding ratios of the areas and lengths (volumes and areas) of the parts into which L divides K or its boundary then in E^2:

$$m^2(2m+1)^{-1}\leqq k\leqq m^2+2m,\quad m(m+2)^{-1}<g<2m+1,$$

and in E^3:

$$m^3(3m^2+3m+1)^{-1}\leqq k\leqq m^3+3m^2+3m,$$
$$m^2(2m^2+4m+1)^{-1}<g<m^2+4m+2.$$

These inequalities are sharp.

H. Busemann.

Hadwiger, Hugo. Minkowskis Ungleichungen und nichtkonvexe Rotationskörper. Math. Nachr. **14** (1955), 377–383 (1956).

An adequate generalization of the so-called "Quermassintegrale" for a non-convex body A in E^n is given by

$$W_i = c_{in}^{-1} \int \chi(A \cap E_i)\, dE_i \quad (0 \leq i \leq n),$$

where E_i transverses the i-dimensional planes in E^n, dE_i is their integral geometric density, and $\chi(A \cap E_i)$ is the Euler characteristic of $A \cap E_i$. It is assumed that $\chi(A \cap E_i)$ is defined for all E_i and that the integral exists as a Riemann integral. The constant c_{in} is defined by $c_{0n}=1$, $c_{in} = \binom{n}{i} (\kappa_{n-1} \cdots \kappa_{n-i})(\kappa_1 \cdots \kappa_i)^{-1}$ for $i \geq 1$, where κ_i is the volume of the i-dimensional unit ball. For convex A the W_i satisfy the inequalities

$$(W_k/\kappa_n)^{1/(n-k)} \geq (W_i/\kappa_n)^{1/(n-i)} \quad (0 \leq i < k \leq n-1),$$

with equality only for spheres. It is known that these inequalities are not generally true, but it is shown here that they remain valid for non-convex bodies of revolution, when each hyperplane normal to the axis of revolution intersects the body in a set homeomorphic to an $(n-1)$-dimensional ball. *H. Busemann.*

Hanner, Olof. Intersections of translates of convex bodies. Math. Scand. **4** (1956), 65–87.

Let K be a convex body with interior points in E^n. Define $I(K)$ as the smallest integer m for which vectors $u_1, \cdots, u_m$ exist with $(K+u_i) \cap (K+u_j) \neq 0$ but

$$\bigcap_{k=1}^{m} (K+u_k) = 0,$$

and put $I(K)=\infty$ if no m exists. It was proved by B. v. Sz. Nagy [Acta Sci. Math. Szeged **15** (1954), 169–177; MR **16**, 507] that $I(K)=\infty$ for parallelepipeds and $I(K) \leq n+1$ otherwise. It is proved here that $I(K)$ takes only the values 3, 4, ∞ and that $I(K)>3$ if, and only if, K is a polyhedron with center such that for any two disjoint faces L_1, L_2 of K two parallel supporting planes P_1, P_2, of K with $P_i \supset L_i$ exist. For each n there is only a finite number of affinely non-equivalent K with $I(K)>3$. The number $I(K)$ does not change if instead of translates $K+u_i$ of K, bodies l_iK+u_i homothetic to K are admitted. *H. Busemann.*

Hadwiger, H.; und Ohmann, D. Brunn-Minkowskischer Satz und Isoperimetrie. Math. Z. **66** (1956), 1–8.

Let A and B be bounded closed sets with positive measures $|A|$, $|B|$ in E^n. Then (Brunn-Minkowski Theorem)

$$(\alpha) \qquad |A+B|^{1/n} \geq |A|^{1/n} + |B|^{1/n}.$$

This implies

$$(\alpha\alpha) \qquad \liminf_{h \to 0+} (|A+hB| - |B|) h^{-1} \geq n|A|^{(n-1)/n} |B|^{1/n}.$$

The equality holds in (α) only when A and B are convex and homo- thetic. Denote by A_e the set of all points of A about which no sphere with positive radius exists whose intersection with$A$ has measure 0. If $A=A_e$ and $B=B_e$ then equality holds in ($\alpha\alpha$) only when A and B are convex and homothetic.

As to the results: the condition for the equality sign in ($\alpha\alpha$) had been discussed previously (under a slighlty weaker condition than $A=A_e$) only in the case where B is assumed to be convex. The method of proof for both (α) and ($\alpha\alpha$) is new. Its most interesting feature is this: Brunn's fundamental idea of dividing the volumes of A and B proportionally by parallel planes, which is usually used for an induction with respect to the dimension, is employed here to pass from certain subsets of A and B to A and B themselves. *H. Busemann.*

Egloff, Werner. Eine Bemerkung zu Cauchy's Satz über die Starrheit konvexer Vielflache. Abh. Math. Sem. Univ. Hamburg **20** (1956), 253–256.

Cauchy's proof of the rigidity of a convex polyhedron requires a lemma concerning plane polygons: If two convex polygons $A_1A_2 \cdots A_n$ and $B_1B_2 \cdots B_n$ are so related that $A_rA_{r+1} = B_rB_{r+1}$ $(r=1, 2, \cdots, n-1)$ and $\angle A_{r-1}A_rA_{r+1} \leq \angle B_{r-1}B_rB_{r+1}$ $(r=2, \cdots, n-1)$, then $A_1A_n \leq B_1B_n$. The author gives a new proof for this lemma, more rigorous than Cauchy's [J. École Polytech. **9** (1813), cahier 16, 87–98 = Oeuvres complètes, sér. 2, t. 1, Gauthier-Villars, Paris, 1905, pp. 26–38] and simpler than that of Steinitz and Rademacher [Vorlesungen über die Theorie der Polyeder, Springer, Berlin, 1934, pp. 60–67]. *H. S. M. Coxeter* (Toronto, Ont.).

Bloh, È. L. On the most dense arrangement of spherical segments on a hypersphere. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 707–712. (Russian)

The author applies the Blichfeldt density method to find an upper bound for the density of packing of equal spherical caps of angular radius θ_*, and compares his results with these given by Chabauty [C. R. Acad. Sci. Paris **236** (1953), 1462–1464; MR **14**, 850]. The argument is based on a result equivalent to Lemma 1 of a paper by the reviewer [Proc. Glasgow Math. Assoc. **2** (1955), 139–144; MR **17**, 523] of which the author is unaware. His density-function differs slightly from that taken by the reviewer, since he takes into account the fact that not more than $n+1$ of the superposed caps of variable density can overlap at a point. His final upper bound for the density of packing is a complicated expression which is smaller by a factor

$$2\left[1+\left\{1+\frac{\tan^2 \theta_m}{n+1}\right\}^{\frac{1}{2}}\right]^{-1}$$

than that obtained by the reviewer; here $\sin \theta_m = 2^{\frac{1}{2}} \sin \theta_*$. A graph is given to illustrate how this estimate varies for different sizes of cap and values of n. On p. 708 the line defining density-function should read

$$\rho(\theta) = \sin^2 \frac{\theta_n}{2} - \sin^2 \frac{\theta}{2}.$$

R. A. Rankin (Glasgow).

Florian, A. Ungleichungen über konvexe Polyeder. Monatsh. Math. **60** (1956), 288–297.

The author continues his discussion [Monatsh. Math. **60** (1956), 130–156; MR **17**, 1235] of inequalities for convex polyhedra in the unit sphere. He introduces a conjecture on the "edge curvature" of such polyhedra and proves it under certain special assumptions.

E. G. Straus (Los Angeles, Calif.).

See also: Freudenthal, p. 591.

Differential Geometry

Gheorghiu, Octavian. La détermination de la loi de transformation des objects différentiels-géométriques de deuxième classe, à deux composantes, en X_1. Com. Acad. R. P. Române 1 (1951), 1017–1020. (Romanian. Russian and French summaries)

Considering geometric objects of the second class and with two components in a one-dimensional space, the author gives an alternative demonstration of some results previously obtained by Penzov [Mat. Sb. N.S. **26(68)** (1950), 161–182; MR **12**, 52]. *L. A. MacColl.*

Gheorghiu, Octavian Em. Un objet géométrique pseudolinéaire de 1-ère classe, à deux composantes. Com. Acad. R. P. Române 2 (1952), 1–4. (Romanian. Russian and French summaries)

Considering an n-dimensional space, the author determines a geometric object having two components which, under a transformation of coordinates

$$\bar{x}^i=\bar{x}^i(x^1, \cdots, x^n),$$

are transformed according to the law

$$\bar{\Omega}_1=\Omega_1+a,\ \bar{\Omega}_2=\Omega_2+(b/a)(e^a-1)e^{\Omega_1}.$$

It is found that, in order for this transformation to have the necessary group property, the parameters a and b must have the values $a=k\log|\Delta|$, $b=ha$, where h and k are constants and Δ denotes the Jacobian $|\partial\bar{x}^i/\partial\bar{x}^j|$. It is shown that this geometric object can be expressed in terms of another, having two components which, under a transformation of coordinates, are transformed linearly.
L. A. MacColl (New York, N.Y.).

★**Haack, Wolfgang. Elementare Differentialgeometrie.** Birkhäuser Verlag, Basel und Stuttgart, 1955. viii+239 pp. 22 francs suisses.

Chapters I, II and IV–VIII of this book are the same as the author's Differential-Geometrie, T. I [Wolfenbütteler Verlagsanstalt, Wolfenbüttel-Hannover, 1948; MR **9**, 612]. To these are added three new chapters III, IX and X. The first two of these re-do much of the standard material on curves and surfaces in Euclidean space as worked out in the other chapters, but this time following the methods of Élie Cartan. That is, a curve or surface, together with tangent three-frames (of orthonormal vectors) is considered as a submanifold of the space of all three frames at all points of Euclidean space. Then the fundamental forms of classical differential geometry are obtained by restricting the equations of structure of this larger manifold to the submanifold. By the end of Chapter X the author has given then a two-fold exposition of classical differential geometry of curves and surfaces, and in quite a clear and concise fashion. The book is intended primarily as a text but could also be useful for anyone wishing to see the relation between the two methods of approach to differential geometry. It uses very little from the theory of partial differential equations, especially through Chapter VIII.

Chapter X however is somewhat different in character from the earlier part of the book. It is a rather detailed and relatively self-contained exposition of the fundamental existence theorem of surface theory, namely that an abstractly given first and second fundamental form satisfying the differentiability conditions can be realized by a surface in Euclidean space; together with material on the closely related subject of bending of surfaces, subjects to which the author has himself contributed. The proofs in this section are carefully worked out so as to require a minimum of previous knowledge on the part of the reader — and yet they are not too long. A special bibliography listing about thirty-five papers on this subject is included. This chapter especially is a useful addition to the original book. *W. M. Boothby.*

Vincensini, P. Sur le problème de la transformation, par déformation, d'un réseau asymptotique en réseau conjugué. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 928–938.

This problem, solved by L. Bianchi [Atti Accad. Lincei. Rend. Cl. Sci. Fis. Mat. Nat (5) **13** (1904), 1° semestre, 6–17, 147–161], is considered from the point of view expressed in one of the author's papers [Ann. Sci. Ecole Norm. Sup. (3) **64** (1947), 197–226; MR **10**, 64]. If a point $P(u, v)$ of a surface S is the center of a sphere of radius $R(u, v)$, which touches the envelope of these spheres at M_1M_2, then the point of intersection I of M_1M_2 with the tangent plane at P form an association (P, I) of the type described in this paper of 1947. The corresponding invariant net on S is given by

$$ds^2-\rho_{11}du^2-2\rho_{12}dudv-\rho_{22}dv^2,$$

where ρ_{ij} are the second covariant derivatives of $\rho=\frac{1}{2}R^2$. It is shown that every asymptotic net is such an invariant net, corresponding to ∞^3 possible associations, provided by the ∞^3 congruences of spheres passing through a point of space. Deformation of S transforms these nets into so-called virtual asymptotic nets, characterized by the applicability relation of Bianchi $\Delta_{22}\rho-\Delta_2\rho+1=K(2\rho-\Delta\rho)$, where the Δ are differential parameters and K is the curvature of S. Then the condition that virtual asymptotic nets will be conjugate is also the condition on the ds^2 which solves the proposed problem.
D. J. Struik (Cambridge, Mass.).

Strel'cov, V. V. On the extension of surfaces. Izv. Akad. Nauk Kazah. SSR. Ser. Mat. Meh. 1956, no. 4(8), 128–140. (Russian)

With all concepts understood in the general sense of A. D. Alexandrov the author proves: Let S lie in E^3 and be homeomorphic to a disk, have non-positive curvature and diameter d. Then S can be extended to a surface S_1, with the same properties such that the length of the boundary curve of S_1 is at most $(2\pi-\omega^-)d/2$ when $\tau^->-\pi$, and at most $(\pi-\omega^--\tau^-)d/2$ when $\tau^-<-\pi$, where ω^- is the total curvature of S and τ^- is the negative part of the total geodesic curvature of the boundary of S.
H. Busemann (Los Angeles, Calif.).

Penzov, Yu. E. On bundles of one-dimensional geometrical objects in an X_1^r of class ≥ 2. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 356–359. (Russian)

In this paper the author considers equivalence classes of bundles of geometric objects defined on a manifold X_n^r of dimension n and class C^r. Let Y be a Hausdorff space acted on effectively by a group G and let $h(\xi)$ be a homeomorphism of the full differential group $\delta^{v,n}$ [for definition see Haantjes and Laman, Nederl. Akad. Wetensch. Proc. Ser. A. **56** (1953), 208–215; MR **15**, 990] onto G depending continuously on $\xi\in X_n^r$. Then if $v\leq r$ and V_i, V_j are coordinate neighborhoods on X_n^r the equations of change of coordinates and their derivatives determine in a natural fashion an element $x_{ji}(\xi)$ of $\delta^{v,n}$ for each $\xi\in V_i\cap V_j$. Then $g_{ji}(\xi)=h(\xi)[x_{ji}(\xi)]$ determine coordinate transfor-

mations of a bundle over $X_n{}^r$ with fibre Y. Such a bundle is called a bundle of differential geometric objects of class v. If G acts transitively on Y the bundle is said to be transitive. With the assumption that $h(\xi)$ is constant the author determines the transitive bundles of class $\geqq 2$ on a one-dimensional base space $X_1{}^r$, those of class one having been determined by Haantjes and Laman. In this special case $\delta^{v,1}$ becomes a Lie group. Four equivalence classes are distinguished, and their coordinate transformations are explicitly given. *W. M. Boothby* (Evanston, Ill.).

Gheorghiu, O. Em. Objets géométriques différentiels de 1ère classe à deux composants en X_1. Acad. R. P. Romîne. Baza Cerc. Şti. Timişoara. Stud. Cerc. Şti. Ser. I. 2 (1955), 21–25. (Romanian. Russian and French summaries)

The purpose of this note is to establish the law of transformation of differential geometrical objects of class C^1 with two components, in a space of one dimension X_1. It is shown that in the transformation formulas occur two strictly monotone but otherwise arbitrary functions and their inverses. *R. Blum* (Saskatoon, Sask.).

Graeub, Werner; und Nevanlinna, Rolf. Zur Grundlegung der affinen Differentialgeometrie. Ann. Acad. Sci. Fenn. Ser. A. I. no. 224 (1956), 23 pp.

Für eine „Grundlegung" der affinen Differentialgeometrie ist naturgemäsz eine Präzisierung der „Grundbegriffe" das Allernotwendigste. Die Verfasser beginnen entsprechend in einem topologischen Raum $\mathscr{M}$, der dem Hausdorffschen Trennungsaxiom genügt. $\mathscr{M}$ wird mit abzählbar vielen Umgebungen $\mathscr{U}$ überdeckt. Ist jede Umgebung $\mathscr{U}$ zu einem Gebiet $\mathscr{U}_x$ eines n-dimensionalen linearen Raumes $\mathscr{L}_x$ homöomorph, so ist $\mathscr{M}$ eine n-dimensionale Mannigfaltigkeit $\mathscr{M}^n$. $\mathscr{L}_x$ heiszt „Parameterraum" und $\mathscr{U}_x$ „Parameterumgebung". Aus zwei Umgebungen $\mathscr{U}$, $\bar{\mathscr{U}}$, deren Durchschnitt nicht leer ist, ergeben sich homöomorphe Beziehungen $\mathscr{U}_x \to M^n \to \bar{\mathscr{U}}_x$ mit zugeordneten Parameterpunkten x, $\bar{x}$, $,\bar{x}$ $\cdots$. — Die Mannigfaltigkeit heiszt differenzierbar, wenn die Parametertransformationen $x \leftrightarrow \bar{x}$ differenzierbar sind. Entsprechend wird mehrfache Differenzierbarkeit erklärt. — Sodann folgt der Aufbau des Vektor- und Tensorkalküls. Dabei bilden die reellen linearen Funktionen in $\mathscr{L}$ den zu $\mathscr{L}$ dualen Raum $\mathscr{L}^*$, welchem die kovarianten Vektoren und Tensoren entspringen. — Der Theorie der Parallelverschiebung wird die Definition des „Verschiebungsoperators" T als Selbsttransformation des Tangentialraumes vorangestellt. Für den Produktweg $l_2 l_1$ zweier Wege l_1 und l_2 wird $T_{l_2 l_1} = T_{l_2} T_{l_1}$ verlangt, für zwei reziproke Wege l und l^{-1} gilt $T_l T_l^{-1} = T_l^{-1} T_l = E$ = identische Transformation. Als Hauptproblem der Theorie der Parallelverschiebung werden die Bedingungen untersucht, unter welchen die Parallelverschiebung vom Wege unabhängig wird. Damit kommen die Verfasser naturgemäsz zu einer Theorie des Krümmungstensors. Er erscheint als ein gewisser Grenzoperator, der (wo er existiert) durch den Verschiebungsoperator T eindeutig bestimmt ist. — In der Theorie der Differentialgleichungen der Parallelverschiebung wird auch der „affine Zusammenhang" als ein Operator Γ eingeführt und gezeigt, dasz Γ kein Tensor ist. Zum Schlusz wird die kovariante Ableitung behandelt. Sie wird als lineare Selbstabbildung des Tangentialraumes $\mathscr{L}_{p_0}$ aufgefasst, die jeder Anfangsrichtung die kovariante Ableitung, die selbst wieder ein Vektor in $\mathscr{L}_{p_0}$ ist, zuordnet. Die kovariante Ableitung eines kontravarianten Vektorfeldes wird so zum gemischten Tensor zweiter Stufe. Wenn man einen Zusammenhang zwischen der kovarianten Differenzierbarkeit eines Vektorfeldes und der Differenzierbarkeit bezüglich der lokalen Parameter aufstellt, kommt man wieder auf die Differentialgleichungen der Parallelverschiebung. *M. Pinl* (Köln).

Sorace, Orazio. Sulle superficie di S_4 aventi cinque iperpiani di Blaschke indipendenti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 20 (1956), 452–456.

Sia F una superficie dello spazio proiettivo S_4. Un S_3 di Blaschke di F è secondo B. Segre [Abh. Math. Sem. Univ. Hamburg 20 (1955), 28–40; MR 17, 1238] un S_3 segato dai piani tangenti di F secondo rette di una congruenza W a falde focali distinte e non degeneri. B. Segre, rispondendo ad una questione posta da W. Blaschke, ha dimostrato che esistono in S_4 superficie dotate di ∞^1 S_3 di Blaschke, inviluppanti un cono quadrico di seconda specie, ed ha determinato tali superficie assegnandone una rappresentazione parametrica con l'introduzione di due funzioni arbitrari di un solo argomento; ed ha inoltre provato che ogni altra superficie di S_4 possiede k S_3 di Blachke linearmente indipendenti ($0 \leqq k \leqq 5$), indicando anzi una classe di superficie aventi il numero massimo finito (e cioè 5) di S_3 di Blaschke. Nella presente Nota l'Autore, seguendo la via indicata da B. Segre, determina tutte le superficie di S_4 aventi cinque S_3 di Blaschke linearmente indipendenti. *D. Gallarati* (Genova).

Atanasyan, L. S. Some special manifolds of a multidimensional affine space. Moskov. Gos. Ped. Inst. Uč. Zap. 71 (1953), 19–44. (Russian)

This is a study of V_n in affine E_m given by

$$r = r(x^1, \cdots, x^n),$$

for which the space E_{n+p} spanned by the vectors r_i, r_{ij} has lower than maximum dimension, e.g. $p=1$ (axial point), $p=2$ (planar point). This involves certain algebraic relations between the tensors $\overset{\alpha}{h}_{ij}$, $\underset{\sigma}{\beta_i{}^h}$, $\underset{\alpha}{\overset{\lambda}{\gamma_i}}$ [Atanasyan, Trudy Sem. Vektor. Tenzor. Anal. 9 (1952), 351–410; MR 14, 796]. The following result is typical. If an exterior orthogonal system of $p' \leqq p$ tensors $\overset{A}{h}_{ij}$ and $p'(p+1)$ vectors $\underset{A}{\overset{\rho}{\gamma_i}}$ ($A = 1, \cdots, p'$, $\rho = p+1, \cdots, m-n$) is called regular, if the h are independent, if from $\overset{A}{\psi} \underset{A}{\overset{\rho}{\gamma_i}} = 0$ follows $\overset{1}{\psi} = \overset{2}{\psi} = \cdots = \overset{p'}{\psi} = 0$ and the dimension l of the region of the tensors h and γ does not exceed p', then the theorem holds: If the rigging of a V_n in E_m with $p < \min(n, m-n)$ consists of the vectors $\underset{\alpha}{\xi}$ ($\alpha = 1, \cdots, p$) inside the E_{n+p} and the $\underset{\rho}{\xi}$ ($\rho = p+1, \cdots, m-n$) outside of it, and the exterior orthogonal system of the h and γ is regular, then the V_n is developable of rank $l \leqq p$. *D. J. Struik.*

Atanasyan, L. S. Manifolds of a particular form imbedded in a centro-affine space. Moskov. Gos. Ped. Inst. Uč. Zap. 71 (1953), 3–17. (Russian)

In this paper certain degenerated manifolds V_n in centro-affine space E_m are discussed with the aid of the formulas

$$r_{ij} = \Gamma_{ij}{}^h r_h + \overset{u}{h}_{ij} \underset{u}{\xi}, \quad \underset{u}{\xi}_j = \underset{u}{\beta}_j{}^h r_h + \underset{u}{\overset{\lambda}{\gamma}}_j \underset{\lambda}{\xi},$$

where $r_i=\partial r/\partial x^i$, r the radius vector $r(x^1, \cdots, x^n)$, and the ξ are $m-n$ "rigging" vectors. There are four sets of integrability conditions, of which one is $\overset{n}{h}_{[ij}\beta^h{}_{l]}=\overset{n}{R}{}^h{}_{ijl}$. Conditions are given that a V_m is an E_m, that it lies in a E_{n+p} in E_m, and that it is a hypersurface of some E_{n+1}, passing through the center of E_m. There is also a discussion of the case that the E_m has its center at infinity. Special attention is given to a theorem by S. A. Pyaseckiĭ [Dissertation, Moscow, 1948]. *D. J. Struik.*

Pyaseckiĭ, S. A. On the differential geometry of a hypersurface on an affine complex space. Moskov. Gos. Ped. Inst. Uč. Zap. 71 (1953), 99–126. (Russian)

Given is a surface $r=r(u^1, \cdots, u^n)$ in a complex equiaffine space A_{n+1} with a normalized "normal" vector N_0 of weight $n+1$ defined by

$$(N_0r_1r_2\cdots r_n)=\lambda=\det|\lambda_{ik}|,\ \lambda_{ik}=(r_{ik}r_1\cdots r_n).$$

Then $r_{ik}=\Gamma_{ik}{}^{\sigma}r_{\sigma}+C_{ik}N_0$, $C_{ik}=\lambda^{-1}\lambda_{ik}$. This expression still allows some freedom: $r_{ik}=\bar{\Gamma}_{ik}{}^{\sigma}r_{\sigma}+C_{ik}\bar{N}$, $\bar{N}=\rho^{\alpha}r_{\alpha}+N_0$, $\bar{\Gamma}_{ik}{}^{\sigma}=\Gamma_{ik}{}^{\sigma}-\rho^{\sigma}C_{ik}$. Then, if

$$\bar{N}_{|k}=\partial_k\bar{N}-(n+1)\bar{\Gamma}_{k\sigma}{}^{\sigma}\bar{N}=C_k{}^{\sigma}r_{\sigma},$$

the fundamental relations can be written in the form $r_{i|k}=C_{ik}N$, $N_{|k}=C_k{}^{\alpha}r_{\alpha}$, $(Nr_1\cdots r_n)=\lambda$. To these are added the integrability relations ($\Gamma^i{}_{[kl]}=0$, $C_{[ik]}=0$):

$$R_{ikl}{}^m=C_{i[k}C^m{}_{l]},\ C_{i[|k|l]}=0,\ C^i{}_{[k|l]}=0,$$

$$C_{[i}{}^{\sigma}C_{k]\sigma}+(n+1)\partial_{[k}\Gamma_{i]}{}^{\sigma}{}_{\sigma}=0.$$

The tensor g_{ik} is that of the Blaschke-Lopšic theory: $g_{ik}=\lambda_{ik}|\lambda|^{-1/(n+2)}$. It is shown how the three forms of L. Berwald $\phi_1=A_{\alpha}du^{\alpha}$, $\phi_2=A_{\beta\alpha}du^{\alpha}du^{\beta}$, $\phi_3=A_{\alpha\beta\gamma}du^{\alpha}du^{\beta}du^{\gamma}$ fit into the theory. Special reference is made to a paper by A. M. Lopšic [Trudy Sem. Vektor Tenzor. Anal. 8 (1950), 273–285; MR 12, 636]. *D. J. Struik.*

Pyaseckiĭ, S. A. On the foundations of the differential geometry of a hypersurface of a centro-affine space. Moskov. Gos. Ped. Inst. Uč. Zap. 71 (1953), 127–153. (Russian)

Given is a hypersurface $r=r(u^1, u^2, \cdots, u^n)$ in a centro-affine space E_{n+1} [for the case $n=2$ see Ya. S. Dubnov and V. N. Skrydlov, Trudy Sem. Vektor. Tenzor. Anal. 8 (1950), 128–143; MR 13, 777]. One method to obtain fundamental equations is to take $r_{ik}=\Lambda_{ik}{}^{\alpha}r_{\alpha}+a_{ik}r$ with

$$\Lambda_{ik}{}^l=(-1)^lV^{-1}(r_{ik}rr_1\cdots r_{l-1}r_{l+1}\cdots r_n),$$

$$a_{ik}=V^{-1}(r_{ik}r_1\cdots r_n),\ V=(rr_1r_2\cdots r_n)\neq0.$$

If $\Lambda_{ilk}{}^m$ is the curvature tensor belonging to $\Lambda_{kl}{}^{\alpha}$, then $\Lambda_{ilk}{}^m=a_{i[k}\delta^m{}_{l]}$, $a_{i[k|l]}=0$, when $r_{i|k}=a_{ik}r$. Another method starts with $b_{ik}=(r_{ik}r_1\cdots r_n)/(rr_1\cdots r_n)$ as basic tensor for the Christoffel symbols in

$$r_{i,k}=r_{ik}-\left\{{\alpha \atop lk}\right\}r_{\alpha}=H_{ik}{}^{\alpha}r_{\alpha}+H_{ik}r,\ H_{ik}=b_{ik}.$$

It is shown how affine and equiaffine surface invariants can be obtained from centro-affine ones and how an affine-invariant rigging can be obtained. Applications are made to affine "hyperspheres" (proper or improper depending on the nature of the intersection of the affine normals) and to the centro-affine characterization of hyperquadrics.

D. J. Struik (Cambridge, Mass.).

See also: Frölicher and Nijenhuis, p. 569; Drinfel'd, p. 583; Florian, p. 595; Ishihara, p. 599; Gheorghiev, p. 616; Altrichter, und Schäfer, p. 621.

Riemannian Geometry, Connections

Goldberg, S. I. On pseudo-harmonic and pseudo-Killing vectors in metric manifolds with torsion. Ann. of Math. (2) 64 (1956), 364–373.

The author considers a compact Riemannian manifold on which is given a metric connection $E_{jk}{}^i$ other than the usual Christoffel one, i.e. the torsion $S_{jk}{}^i=\frac{1}{2}(E_{jk}{}^i-E_{kj}{}^i)$ is not assumed to vanish. Following Bochner and Yano [Curvature and Betti numbers, Princeton, 1953; MR 15, 989] a vector with components ξ^i is pseudo-harmonic if $\xi^i{}_{|j}=\xi_{j|i}$ and $\xi^i{}_{|i}=0$, differentiation being with respect to $E_{jk}{}^i$, and pseudo-Killing if $\xi_{j|k}+\xi_{k|j}=0$. Then applying Green's Theorem the author obtains some matrices, involving the curvature and torsion of the connection, whose definiteness would imply the non-existence of pseudo-harmonic or pseudo-Killing vectors. If the Ricci curvature E_{jk} of the connection has the property that $E_{jk}+E_{kj}=\lambda g_{jk}$, g_{jk} being the metric tensor, then the author calls the space pseudo-Einstein. In this special case he obtains the most complete results. For example he proves that in a compact pseudo-Einstein manifold with antisymmetric torsion tensor S_{ijk} and scalar curvature $E\geqq0$, an ordinary harmonic vector must have vanishing covariant derivative with respect to the Christoffel symbols. Moreover if $\|S_j{}^{rs}S_{krs}\|$ has rank equal to the dimension of the manifold, or if the scalar curvature is positive then there exists no ordinary harmonic vector field. Hence the first Betti number is zero. *W. M. Boothby* (Evanston, Ill.).

Černyšenko, V. M. Tensor characteristics of certain types of pairs of congruences of a Riemann space. Trudy Sem. Vektor. Tenzor. Anal. 10 (1956), 279–283. (Russian)

Toutes les considérations sont locales. Dans un espace de Riemann R_n, l'ensemble formé par une congruence de variétés à p dimensions et une congruence de variétés à q dimensions est appelé (p, q)-paire; une telle paire est dite holonome si on peut choisir un système de coordonnées $u^1, \cdots, u^n$ tel que les variétés des deux congruences aient pour équations $u^{p+1}=c^{te}$, $u^{p+2}=c^{te}$, $\cdots$, $u^n=c^{te}$ et $u^1=c^{te}$, $u^2=c^{te}$, $\cdots$, $u^p=c^{te}$, $u^{p+q+1}=c^{te}$, $\cdots$, $u^n=c^{te}$ respectivement. Une paire holonome est dite „métriquement semi-tchebychevienne" si la correspondance déterminée entre deux variétés quelconques de la première congruence par les variétés de la seconde conserve les hypervolumes (à p dimensions); elle est „métriquement tchebychevienne" si la même chose reste vraie lorsqu'on échange les deux congruences. Une paire quelconque est dite „semi-tchebychevienne au sens du transport" si un vecteur tangent à une variété de la première congruence reste tangent aux variétés de la première congruence par transport parallèle le long d'une variété de la seconde congruence; elle est „tchebychevienne au sens du transport" si la même chose reste vraie lorsqu'on échange les deux congruences. L'auteur donne, sous forme tensorielle, les conditions pour qu'une (p, q)-paire soit semi-tchebychevienne ou tchebychevienne, dans l'un ou l'autre sens.

J. L. Tits (Bruxelles).

Lichnerowicz, André. Sur les transformations affines des variétés riemanniennes. C. R. Acad. Sci. Paris **242** (1956), 1568–1570.

This paper amplifies some results of Kobayashi [Nagoya Math. J. **9** (1955), 39–41; MR **17**, 892], Nomizu and Yano [C. R. Acad. Sci. Paris **237** (1953), 1308–1310; Nagoya Math. J. **9** (1955), 43–56; MR **15**, 468; **17**, 891] [Ann. of Math. (2) **55** (1952), 38–45; MR **13**, 689] concerning the group of affine transformations, the group of isometries, and the de Rham decomposition of a complete Riemannian manifold. *W. Ambrose* (Cambridge, Mass.).

Ishihara, Shigeru. Homogeneous Riemannian spaces of four dimensions. J. Math. Soc. Japan **7** (1955), 345–370.

The methods of E. Cartan [Leçons sur la géométrie des espaces de Riemann, 2ième ed., Gauthier-Villars, Paris, 1946; MR **8**, 602] to construct homogeneous n-dimensional Riemannian spaces are used here to deal with one of the 'exceptional cases' ($n=4$) in some theorems by Yano [Trans. Amer. Math. Soc. **74** (1953), 260–279; MR **14**, 688]. — Let $M=G/g$ be homogeneous Riemannian of dimension 4, connected and simply connected. Then M is homeomorphic to one of the following: E^4, S^4, $P(C, 2)$, $S^2\times S^2$, $E^1\times S^3$, $E^2\times S^2$. The local structures of M and G (dim $G\geqq 6$) are determined as follows: (1) if $r=10$, M is of constant sectional curvature; $G\cong R(5)$, $L(5)$, or $\mathfrak{M}(4)$; (2) if $r=8$, M is a two-dimensional Kählerian space with constant holomorphic curvature and $G\cong SU(3)$, $S\mathfrak{L}(3)$ or $\mathfrak{M}_H(2)$; (3) if $r=7$, M is the product of a straight line and a 3-dimensional Riemannian space with constant sectional curvature, or M is a 4-dimensional Riemannian space with constant negative sectional curvature; and in this case $G\cong A_1\times R(4)$, $A_1\times L(4)$, $A_1\times\mathfrak{M}(3)$, a subgroup of $L(5)$ or $S\mathfrak{M}_H(2)$; if $r=6$, M is a product of two 2-dimensional Riemannian spaces, each of constant curvature, and $G\cong R(3)\times R(3)$, $R(3)\times L(3)$, $R(3)\times\mathfrak{M}(2)$, $L(3)\times L(3)$, $L(3)\times\mathfrak{M}(2)$, or $\mathfrak{M}(2)\times\mathfrak{M}(2)$. — Of Yano's theorems, one becomes for $n=4$: "There is no group of motions of order 9. If there is a group of motions of order 8, then it is transitive, and the space is Kählerian with constant holomorphic sectional curvature." The others remain true, and are included in the above results. If G is compact, the following cases occur: $r=10$, $G\cong R(5)$, M is $C(+,4)$; $r=9$ does not occur; $r=8$, $G\cong SU(3)$, M is $K(+,2)$; $r=7$, $G\cong A_1\times R_4$, M is locally $V^1\times C(+,3)$; $r=6$, $G\cong R(3)\times R(3)$, M is locally $C(+,2)\times C(+,2)$; $r=5$, $G\cong A_2\times R(3)$, M is locally $C(0,2)\times C(+,2)$; $r=4$, $G\cong T^4$, M is flat; or $G\cong A_1\times R(3)$, M is locally $V^1\times C(+,3)$.

Notation: A_r is the r-dimensional vector group over the reals; $C(+,n)$ is a Riemannian space with positive constant curvature; $\mathfrak{L}(n)$ is the Lorentz group in n variables; $\mathfrak{M}(n)$ is the group of proper motions in E^n; $\mathfrak{M}_H(n)$ is the subgroup of M whose rotation parts are elements of $U(n)$; V^n is the n-fold product of a real line.

A. Nijenhuis (Seattle, Wash.).

Obata, Morio. On n-dimensional homogeneous spaces of Lie groups of dimension greater than $n(n-1)/2$. J. Math. Soc. Japan **7** (1955), 371–388.

Let G be a connected Lie group of dimension r, and H a compact subgroup of dimension $r-n$ $(0<n\leqq r)$. G is almost effective on G/H; i.e. H contains a discrete normal subgroup of G. If $n\geqq 2$, and $r\geqq n(n+1)/2$, then G/H is (1) a Riemannian space of positive constant curvature whose universal covering space is a sphere; (2) a Riemannian space of constant negative curvature homeomorphic to E^n; (3) locally flat Riemannian space homeomorphic to E^n. If $n(n-1)/2<r<n(n+1)/2$; $n\geqq 4$, then $r=\frac{1}{2}(n-1)+1$, and G/H is one of the following: $C(0,1)\times C(+,n-1)$; $C(0,1)\times C(-,n-1)$, $C(0,n)$, $C(-,n)$; topologically $E^1\times S^{n-1}$ (if simply connected), E^n or $S^1\times E^{n-1}$, E^n or $S^1\times E^{n-1}$, E^n respectively.

The proofs are based on an analysis of some subgroups of rotation groups and Lorentz groups. (Notations as in the paper reviewed above.) *A. Nijenhuis.*

Ishihara, Shigeru. Groups of projective transformations on a projectively connected manifold. Jap. J. Math. **25** (1955), 37–80 (1956).

Let M be an n-dimensional manifold with a projective connection, and Γ a Lie group of projective transformations of M. This paper is mainly concerned with the determination of M and Γ when dim $\Gamma\geqq n^2+5$. Among other results, the author has proved the following:

I. Suppose that Γ is effective and dim $\Gamma\geqq n^2+5$, and $n\geqq 3$. Then M is locally flat and (i) dim $\Gamma=n^2+2n$, n^2+n or n^2+n-1 when $n\geqq 6$; (ii) dim $\Gamma=n^2+2n$ or n^2+n when $n\geqq 5$; (iii) dim $\Gamma=n^2+2n$ when $n=3, 4$.

II. Suppose that M has non-trivial torsion. Then dim $\Gamma\leqq n^2$.

III. If dim $\Gamma=n^2+2n$, then M is either homeomorphic with a sphere or a real projective space.

The groups of projective transformations of affinely connected spaces (with or without torsion) are also discussed. *H. C. Wang* (New York, N.Y.).

Cossu, Aldo. Una particolare classe di connessioni tensoriali. Rend. Mat. e Appl. (5) **15** (1956), 190–210.

The reviewer has introduced [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **1** (1946), 478–482; MR **8**, 404] the notion of "tensor-connexion", i.e. a connexion for a tensor not based on the notion of connexion for vectors (or affine connexion). For a double contravariant tensor field, for instance, $\xi^{ik}(x)$ the absolute differential is determined by

$$D\xi^{ik}=d\xi^{ik}+L^{ik}_{rst}\xi^{rs}dx^t$$

and the fundamental equations for a change of allowable coordinates $x^i\leftrightarrow x^{i'}$ are

$$L^{i'k'}_{r's't'}=$$
$$L^{ik}_{rst}x_i{}^{i'}x_k{}^{k'}x_{r'}{}^{r}x_{s'}{}^{s}x_{t'}{}^{t}-x_{rt}{}^{i'}\delta_{s'}{}^{k'}x_{r'}{}^{r}x_{t'}{}^{t}-x_{st}{}^{k'}\delta_{r'}{}^{i'}x_{s'}{}^{s}x_{t'}{}^{t},$$

where $x_i{}^{i'}=\partial x^{i'}/\partial x^i$, $x_{rt}{}^{i'}=\partial^2x^{i'}/\partial x^r\partial x^t$, $\cdots$. Similar laws can be given for covariant or mixed tensor.

The new geometric objects can be expressed in terms of affine connexions and tensor intrinsically determined by them. It is the purpose of this paper to determine the geometric meaning of these related objects and the curvature properties of tensors connexions (to be noticed, transformations, geometrically characterized, keeping the curvature tensor invariant). A detailed study is devoted to tensor connexions L::. (for doubly covariant or contravariant tensor) or E::. (for mixed tensors) of the following types:

$$L^{ik}_{rst}=M^i_{rt}\delta^k{}_s+N^k_{st}\delta_r{}^i,\quad E^{is}_{rkt}=M^i_{rt}\delta_k{}^s-N^i_{kt}\delta_r{}^i$$

M:., N:. being the components of affine connexions.

E. Bompiani (Rome).

See also: García, p. 628.

Algebraic Geometry

Turri, Tullio. Su casi limiti di trasformazioni involutorie costruite mediante le quadriche per sei punti. Rend. Sem. Fac. Sci. Univ. Cagliari **25** (1955), 134–136 (1956).

È noto che la superficie dei punti uniti dell'involuzione I_2 determinata nell'S_3 dalle quadriche passanti per sei punti A_i è generalmente una F^4 rappresentabile parametricamente mediante funzioni theta di due argomenti. Se la configurazione dei punti A_i si particolarizza, la F^4 può riuscire razionale: l'Autore indica due casi semplici in cui tale fatto si verifica. *D. Gallarati* (Genova).

Abhyankar, Shreeram. Simultaneous resolution for algebraic surfaces. Amer. J. Math. **78** (1956), 761–790.

Les travaux de Zariski et de l'auteur ont montré que tout corps K de fonctions algébriques à 2 variables sur un corps de base parfait k admet un modèle non singulier. Soit alors K' une extension algébrique finie de K. Si une valuation v' de dimension 0 de K'/k est uniformisable, elle est uniformisable sur un modèle V' de K' qui est la normalisation projective d'un modèle de K. Si k est algébriquement clos et de caractéristique 0 et si v' est à valeurs rationnelles, il existe un modèle projectif non singulier V de K tel que le centre de v' sur la normalisation projective V' de V dans K' soit un point simple; si de plus K' est une extension galoisienne de K, il existe un modèle projectif normal de K dont la normalisation dans V' est non-singulière. Si k est un corps parfait de caractéristique $p\neq 0$ et si K' est une extension cyclique quadratique (pour $p\neq 2$) ou cubique (pour $p\neq 3$) de K, il existe un modèle non singulier V de K dont la normalisation V' dans K' est non-singulière; de théorème de "résolution simultanée des singularités" ne s'étend pas aux extensions cycliques de degré premier >3. Ce mémoire contient aussi des lemmes sur les extensions kummeriennes des anneaux factoriels, et des généralisations partielles aux variétés de dimension 3. *P. Samuel.*

★ **Néron, André. Arithmétique et classes de diviseurs sur les variétés algébriques.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 139–154. Science Council of Japan, Tokyo, 1956.

L'auteur précise, dans le cas des corps de fonctions, certains points de la théorie des „distributions" de Weil. Il en déduit une simplification de la méthode de descente infinie employée par lui pour montrer que $G(V)/G_a(V)$ est un groupe abélien de type fini. *P. Samuel.*

★ **Nakai, Yoshikazu. Some results in the theory of the differential forms of the first kind on algebraic varieties.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 155–173. Science Council of Japan, Tokyo, 1956.

L'auteur établit d'abord quelques résultats reliant les formes différentielles sur une variété projective normale V, les formes différentielles sur une hypersurface projection générique V' de V, et le conducteur de l'anneau de coordonnées affines $\mathfrak{o}'$ de V' dans l'anneau de coordonnées affines $\mathfrak{o}$ de V. Il en déduit le résultat suivant: si $d^0V<2+\dim(V)$, alors le genre géométrique p_g de V est nul; si $d^0V=2+\dim(V)$ on a $p_g\leq 1$ et, si $p_g=1$, V est une hypersurface. Etablissement de conditions, portant sur les coéfficients d'une forme différentielle ω, qui sont nécessaires pour que ω soit de première espèce; ces conditions sont suffisantes si V est non-singulière, ou si ω est de degré maximum $r=\dim(V)$. Etude de la trace ω_C d'une forme différentielle $\bar{\omega}$ de degré $r-1$ de V^r sur une section hyperplane générique C. Etude des formes différentielles $\bar{\omega}$ de degré $r-1$ sur C pour lesquelles il existe sur V une forme ω de degré r telle que $(\omega)+C\geq 0$ et que $\mathrm{Res}_C(\omega)=\bar{\omega}$; moyennant une conjecture sur „l'indépendance" des dérivées partielles d'un polynôme homogène définissant l'hypersurface projection générique d'une variété non singulière (dont le degré n'est pas multiple de la caractéristique), on montre que (si le degré de V n'est pas un tel multiple) aucune forme $\bar{\omega}$ de première espèce sur C peut être du type précédent. *P. Samuel.*

Nagata, Masayoshi. A general theory of algebraic geometry over Dedekind domains. I. The notion of models. Amer. J. Math. **78** (1956), 78–116.

Ce mémoire est le premier d'une série consacrée à la géométrie algébrique sur un anneau de Dedekind (le cas classique d'un corps étant un cas particulier). Rappel très précis des résultats d'Algèbre locale supposés connus. Démonstration du th. de normalisation pour les anneaux d'intégrité de la forme $k[x_1, \cdots, x_n]$ et généralisation de celui ci aux anneaux de la forme $\mathfrak{o}=I[x_1, \cdots, x_n]$ où I est un anneau dont aucun élément $\neq 0$ n'est diviseur de zero dans $\mathfrak{o}$. On désigne désormais par I un anneau de Dedekind tel que la clôture intégrale de I dans n'importe quelle extension finie du corps des fractions K de I soit un I-module de type fini; on appelle algèbre affine tout anneau d'intégrité de la forme $\mathfrak{o}=I[x_1, \cdots, x_n]$, et corps de fonctions le corps des fractions d'un tel anneau; on appelle localité tout anneau de fractions $P=\mathfrak{o}_\mathfrak{p}$ où $\mathfrak{o}$ est une algèbre affine et $\mathfrak{p}$ un idéal premier de $\mathfrak{o}$; si L est le corps d'effraction de P, on dit que P est une localité de L. Les résultats classiques de la théorie de la dimension sont valables pour les localités. Toute localité P est analytiquement non-ramifiée, et le complété d'une localité normale (=intégralement close) est un anneau d'intégrité normal; la clôture intégrale P' d'une localité P est un P-module de type fini. La clôture intégrale $\mathfrak{o}'$ d'une algèbre affine $\mathfrak{o}$ est une algèbre affine et un $\mathfrak{o}$-module de type fini.

Etant donné un corps de fonctions L (sur I), on appelle place de L sur I tout anneau de valuation de L qui contient I. On dit qu'un anneau local A d'idéal maximal $\mathfrak{m}$ domine un anneau local A' d'idéal maximal $\mathfrak{m}'$ si $A\supset A'$ et si $\mathfrak{m}'=\mathfrak{m}\cap A$; deux localités P, P' de L sont dites apparentées s'il existe une place de L sur I qui domine P et P'. On appelle modèle (ou schéma) affine de L tout ensemble A de localités de L pour lequel il existe une algèbre affine $\mathfrak{o}$ (déterminée de façon unique par A) telle que L soit le corps des fractions de $\mathfrak{o}$ et que A soit l'ensemble des anneaux $\mathfrak{o}_\mathfrak{p}$ ($\mathfrak{p}$ premier). Un modèle (ou schéma) de L est un ensemble M de localités de L qui est réunion finie de modèles affines et tel que deux localités distinctes de M ne soient jamais apparentées; on dit qu'un modèle M de L est complet si toute place de L sur I domine un élément P de M. Notion de modèle projectif, et construction de ceux-ci comme réunions finies de modèles affines; ils sont complets. Une localité P' est appelée une spécialisation d'une localité P si P est un anneau de fractions de P'; notons $M(P)$ l'ensemble des spécialisations d'une localité P qui sont dans le modèle M; étude de ces notions. Etant données deux localités P, P' de L, on appelle joint de P et P' l'ensemble $J(P, P')$ des anneaux de fractions de $P[P']$ qui dominent P et P'; le joint de deux modèles M, M' est la réunion des $J(P, P')$ où $P\in M$ et $P'\in M'$; c'est un modèle, qui est complet (resp. projectif) si M et M' le sont. Etant donné un modèle M d'un corps de fonctions L et

une extension algébrique finie L' de L, notons $N(P, L')$ $(P \in M)$ l'ensemble des anneaux de fractions de la fermeture intégrale de P dans L' par rapport à ses idéaux maximaux; la réunion $N(M, L')$ des $N(P, L')$ $(P \in M)$ est un modèle de L', appelé modèle normal dérivé de M dans L'; tout modèle normal (c.à.d. dont toutes les localités sont normales) de L' qui domine M domine aussi $N(M, L')$ si M est complet (resp. projectif), il en est de même de $N(M, L')$. L'auteur introduit alors la notion de partie irréductible d'un modèle M, et la topologie de Zariski de M (les fermés sont les réunions finies de parties de la forme $M(P)$); celle ci jouit des propriétés usuelles, et coïncide avec la topologie classique sur l'ensemble des idéaux premiers de $\mathfrak{o}$ lorsque M est le modèle affine associé à l'algèbre affine $\mathfrak{o}$; pour qu'une partie M' d'un modèle M de L soit un modèle de L il faut et il suffit qu'elle soit ouverte; l'intersection de deux modèles de L est un modèle de L. Tout fermé irréductible F d'un modèle M est en correspondance biunivoque avec un modèle (obtenu en appliquant à chaque $P \in F$ l'homomorphisme canonique de la localité "générique" de F sur son corps quotient); ce modèle est dit induit par M sur F. Soient M un modèle de L, I' un anneau de Dedekind qui soit un anneau de fractions d'une algèbre affine contenue dans L; l'ensemble des localités $P \in M$ qui contiennent I' est un modèle de L sur I', appelé modèle réduit de L sur I'. Lorsque I est un corps et L une extension régulière de I, tout modèle M de L définit une Variété Abstraite (au sens de Weil) admettant L pour corps de fonctions rationnelles, et inversement. Le présent mémoire est remarquablement clair et bien rédigé. *P. Samuel.*

Weil, André. The field of definition of a variety. Amer. J. Math. 78 (1956), 509–524.

Soient k_0 un corps, k une extension algébrique séparable finie de k_0, $\bar{k}$ sa clôture algébrique, V une variété définie sur k (resp. une variété isomorphe à un ouvert d'une variété projective); si, pour chaque couple (s, t) d'éléments de l'ensemble I des k_0-isomorphismes de k dans $\bar{k}$, il existe une correspondance birationnelle (resp. birégulière) f_{ts} définie sur une extension algébrique séparable de k_0 entre les conjuguées V^s et V^t de V, si l'on a $f_{tr}=f_{ts}\circ f_{sr}$ pour tous $r, s, t \in I$ et si f_{ts} est „covariants" par tous les k_0-automorphismes de $\bar{k}$, alors il existe une variété V_0 définie sur k_0 et une correspondance birationnelle (resp. birégulière) f définie sur k entre V_0 et V telle que l'on ait $f_{ts}=f^t\circ(f^s)^{-1}$ pour tous $s, t \in I$. D'autre part soient k un corps, $k(t)$ une extension régulière de k, $V(t)$ une variété définie sur $k(t)$; si, pour toute paire t, t' de points génériques indépendants du lieu de t sur k, il existe une correspondance birationnelle (resp. birégulière) $f_{t't}$ entre $V(t)$ et $V(t')$ définie sur $k(t, t')$ et telle que $f_{t''t}=f_{t''t'}\circ f_{t't}$, alors il existe une variété V définie sur k et une correspondance birationnelle (resp. birégulière) f_t définie sur $k(t)$ entre V et $V(t)$ telle que $f_{t't}=f_{t'}\circ(f_t)^{-1}$. Conditions pour qu'une variété soit plongeable sur un corps donné dans un espace affine ou projectif. Les théorèmes ci-dessus de „descente du corps de définition" permettent de simplifier les démonstrations de plusieurs résultats de l'auteur, de S. Lang et de W. L. Chow.

P. Samuel.

★**Taniyama, Yutaka. Jacobian varieties and number fields.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 31–45. Science Council of Japan, Tokyo, 1956.

Soit A une variété abélienne. Nous noternons $\mathfrak{A}(A)$ son anneau d'endomorphismes (dont le groupe additif est libre de type fini), $\mu\to\mu'$ son antiautomorphisme involutif, et $\mathfrak{A}_0(A)$ la Q-algèbre $\mathfrak{A}(A)\otimes Q$. Etude des sous corps commutatifs R_0 de $\mathfrak{A}_0(A)$ qui sont globalement invariants par cet antiautomorphisme; ce sont des corps totalement réels ou des extensions quadratiques totalement imaginaires de corps totalement réels. Construction, pour tout idéal $\mathfrak{b}$ de $R_0\cap\mathfrak{A}(A)$, d'une variété abélienne $A_{\mathfrak{b}}$; celle ci est isogène à A et ne dépend que de la classe de $\mathfrak{b}$. Lorsque $[R_0:Q]=2\cdot\dim(A)$ et que A est définie sur un corps de nombres k contenant la clôture galoisienne K' de R_0 sur Q, l'étude des réductions modulo $\mathfrak{p}$ de A permet de calculer la fonction $\zeta_A(s)$ au moyen de fonctions L à „Grössencharaktere" de k; ceci généralise des résultats de Deuring [voir ci-dessous] et de Weil [Trans. Amer. Math Soc. 73 (1952), 487–495; MR 14, 452]. Si K est une extension quadratique totalement imaginaire d'un corps totalement réel K_0 de degré g sur K, et moyennant une hypothèse sur les isomorphismes de K_0 dans C, l'étude des variétés abéliennes correspondant aux classes d'idéaux de K permet de déterminer les extensions abéliennes non ramifiées de K, et de donner une forme explicite à la loi de réciprocité d'Artin. *P. Samuel* (Clermont-Ferrand).

★**Deuring, Max. On the zeta-function of an elliptic function field with complex multiplications.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 47–50. Science Council of Japan, Tokyo, 1956.

Soient k un corps de nombres algébriques, K un corps de fonctions elliptiques sur k. Un diviseur premier $\mathfrak{p}$ de k est dit régulier s'il existe un diviseur premier $\mathfrak{p}'$ de K étendant $\mathfrak{p}$ tel que le corps $K/\mathfrak{p}'$ obtenu par réduction modulo $\mathfrak{p}'$ soit un corps de fonctions elliptiques sur les corps fini $k/\mathfrak{p}$; alors $\mathfrak{p}'$ est unique; notons dans ce cas $\zeta(s, K, \mathfrak{p})$ la fonction ζ de $K/\mathfrak{p}'$. Les diviseurs premiers non réguliers $\mathfrak{q}$ de k sont en nombre fini; pour un tel diviseur on pose $\zeta(s, K, \mathfrak{q})$ égal à la fonction ζ de l'extension transcendante simple de $k/\mathfrak{q}$. On pose $\zeta(s, K)=\prod\zeta(s, K, \mathfrak{p})$ (produit étendu à tous les diviseurs premiers $\mathfrak{p}$ de k, réguliers ou non). Alors $\zeta(s, K)$ s'exprime simplement au moyen de la fonction ζ du corps k et d'une fonction L déduite d'un „Grössencharakter" qui est défini exactement modulo son conducteur. *P. Samuel.*

★**Chevalley, Claude. Plongement projectif d'une variété de groupe.** Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955, pp. 131–138. Science Council of Japan, Tokyo, 1956.

Soit V_1 une variété à loi de composition normale définie sur un corps K. Alors V_1 contient un ouvert affine V (définie sur K) qui est, au sens de Weil, un morceau d'un groupe G défini sur la clôture algébrique de K. Il existe alors un diviseur positif D de V rationnel sur K tel que l'application $s\to sD\cap V$ $(s \in G)$ soit injective; en associant à s le point de Chow de $sD\cap V$ (considéré comme trace sur V d'un diviseur de l'espace projectif ambiant de V), on obtient un plongement de G comme ouvert d'une variété projective. La démonstration d'existence de D comprend, entre autres, les étappes suivantes: construction pour tout $s\neq1$ donné dans G, d'un diviseur positif D_s sur V tel que $D_s\neq sD_s\cap V$; utilisation du fait que, pour tout diviseur D' sur V, l'ensemble des $s \in G$ tels que $sD'\cap V=D'$ est fermé; construction de D par additions successives de diviseurs. *P. Samuel.*

Matsusaka, T. The criteria for algebraic equivalence and the torsion group. Amer. J. Math. **79** (1957), 53–66.

Soit V une variété projective non singulière, $G_n(V)$ (resp. $G_a(V)$, $G_l(V)$) le groupe des diviseurs numériquement (resp. algébriquement, linéairement) équivalents à 0 sur V. L'auteur montre que $G_n(V)/G_a(V)$ est un groupe fini; c'est donc le groupe de torsion du groupe de Néron-Severi de V; ce groupe de torsion est un invariant birationnel dans la classe des variétés non singulières, mais non dans la classe des variétés normales (contre exemple fourni par la variété de Kummer, quotient d'une variété abélienne par la relation $x+y=0$). Des résultats préliminaires sur les critères d'équivalence permettent de se ramener au cas d'une surface; parmi ceux ci figure le fait que, si W est une sous variété d'une variété V, X un diviseur sur V, X' une spécialisation de X sur $k(V, W)$ tels que $X\cdot W$ mais non $X'\cdot W$ soit défini, alors toutes les spécilalisations de $X\cdot W$ prolongeant $X\to X'$ sont des diviseurs linéairement équivalents sur V. Soit alors V une surface non singulière; on montre que deux numériquement équivalents X, X' ont même genre arithmétique virtuel; le fait que $G_n(V)/G_a(V)$ est fini s'en déduit au moyen de l'inégalité de Riemann-Roch. *P. Samuel.*

See also: Abhyankar, p. 556.

NUMERICAL ANALYSIS

Numerical Methods

Keeping, E. S. Note on Wald's method of fitting a straight line when both variables are subject to error. Biometrics 12 (1956), 445–448.

Wald's presentation of a method for fitting straight lines with both variables subject to error using groupings of the ordered observations requires a limitation on the size of the error $\varepsilon_i=x_i-X_i$ (with x_i the observation) in addition to the formal assumptions initially given. The author states the additional assumption in the form "the ε_i are small enough so that the ordering of the observations according to increasing x_i and according to X_i are not substantially different." He shows by illustration that the violation of this assumption may lead to unsatisfactory results such as $\hat{\sigma}_\varepsilon^2<0$ and cautions against the use of this method when the additional assumption is not satisfied. *P. S. Dwyer* (Ann Arbor, Mich.).

Mackenzie, J. K. The estimation of an orientation relationship. Acta Cryst. **10** (1957), 61–62.

A numerical method is given for the determination of an orientation relationship with high accuracy. It is essentially a least-squares method. *W. Nowacki.*

Epstein, Leo F.; and French, Nancy E. Improving the convergence of series: application to some elliptic integrals. Amer. Math. Monthly **63** (1956), 698–704.

The authors describe a method to improve the convergence of an absolutely convergent power series. They write

$$f(x)=\sum_{n=0}^{\infty}\alpha_m x^n=\sum_{n=0}^{\infty}\alpha_n^* x^n+\sum_{n=0}^{\infty}(\alpha_n-\alpha_n^*)x^n,$$

where α_n^* is some conveniently chosen function of n asymptotic with α_n and for which the first sum on the right above is known and where the second sum on the right converges much more rapidly than that on the left. The method is applied to the evaluation of the incomplete elliptic integrals of the first and second kinds, $F(k, \phi)$ and $E(k, \phi)$, as well as the corresponding complete elliptic integrals, $K(k)$ and $E(k)$. High accuracy is obtained using as few as two or three terms of the second series on the right. A further example is given in which the number of terms required for 10-place accuracy is reduced from about 10^5 to 20. *J. G. Herriot* (Stanford, Calif.).

Papoulis, Athanasios. A new method of inversion of the Laplace transform. Quart. Appl. Math. **14** (1957), 405–414.

The author wishes to invert the Laplace transform $R(p)=\int_0^\infty e^{-pt}r(t)dt$ by using the values $R(a+k\sigma)$. First method: Putting $e^{-\sigma t}=\cos\theta$ gives

$$\sigma R[(2k+1)\sigma]=\int_0^{\pi/2}\{\cos^{2k}\theta\sin\theta\}r(-\sigma^{-1}\log\cos\theta)d\theta.$$

The sine coefficients of r can then be found recursively from the values on the left. Some numerical examples are presented. Second method: Put $e^{-\sigma t}=x$, then

$$\sigma R[(2k+1)\sigma]=\int_0^1 x^{2k}r(-\sigma^{-1}\log x)dx.$$

The Legendre coefficients of r can then be found from the values on the left. Finally the author uses Laguerre polynomials to determine $r(t)$ from $R^{(k)}(0)$; this, however, has been done before [Widder, Duke Math. J. **1** (1935), 126–136; Shohat, ibid. **6** (1940), 615–626; MR **2**, 98]. *R. P. Boas, Jr.* (Evanston, Ill.).

Hellman, S. K.; Habetler, George; and Babrov, Harold. Use of numerical analysis in the transient solution of two-dimensional heat-transfer problem with natural and forced convection. Trans. A.S.M.E. **78** (1956), 1155–1161.

The authors obtain a system of finite difference equations giving flow rates and temperatures at time $t+\Delta t$ in terms of those at time t, the physical system being a metallic structure with many flow channels. In appendices they obtain a bound for the truncation error, and show that the system is numerically stable if Δt does not exceed a certain bound which they obtain. *A. S. Householder* (Madison, Wis.).

★ **Nehari, Zeev. On the numerical solution of the Dirichlet problem.** Proceedings of the conference on differential equations (dedicated to A. Weinstein), pp. 157–178. University of Maryland Book Store, College Park, Md., 1956.

The author presents four approaches to the numerical solution of the Dirichlet problem, all utilizing expansions in suitable orthonormal sets of functions. For the region R with boundary B, let H be the set of all functions with finite Dirichlet integral. Then $H=H_0+H_1$ is the direct sum of the subspace H_0 of functions vanishing on B (in a generalized limit sense), and the subspace H_1 of functions harmonic in R. For fixed $\eta\in R$, let $H(\eta)$ be the subset of H_1 subject to the additional restriction $u(\eta)=0$, and let $\{u_\nu\}$ be a complete orthonormal set in $H(\eta)$, using the Dirichlet integral $D[u, v]$ for the inner product. The Bergman kernel function is then

$$k(z, \zeta)=\sum_{\nu=1}^{\infty}u_\nu(z)u_\nu(\zeta).$$

The first approximation uses (essentially) the Fourier coefficients $a_\nu = D[u, u_\nu]$. Let $k_n(z, \zeta) = \sum_{\nu=1}^{n} u_\nu(z) u_\nu(\zeta)$, and consider (essentially)

$$D[u(z) - \sum_{\nu=1}^{n} a_\nu u_\nu(z),\ k(z, \zeta) - k_n(z, \zeta)].$$

In this way one obtains (essentially)

$$[u(\zeta) - \sum_{\nu=1}^{n} a_\nu u_\nu(\zeta)]^2 \leqq [D[u] - \sum_{\nu=1}^{n} a_\nu^2][k(\zeta, \zeta) - k_n(\zeta, \zeta)].$$

But the estimation of the first factor on the right is well known, [J. B. Diaz, Proc. Symposium on Spectral Theory and Differential Problems, Oklahama Agric. Mech. Coll. Stillwater, Okla., 1951, pp. 279–289; MR 13, 235], and the second factor can be estimated by using the fact that $k(\zeta, \zeta)$ is a monotonic function of domain and hence by enclosing ζ in a circle contained entirely within R. This method leads also to estimates on the first-order partial derivatives of u.

The second method depends upon the relation

$$|k(z, \zeta)|^2 / k(\zeta, \zeta) = -(1/2\pi) \partial g(z, \zeta)/\partial n,$$

where k is the Szegö kernel function and g is the usual Green's function. The Szegö kernel function, like the Bergman kernel function, can be approximated by a finite sum of orthonormal functions.

The third method depends upon the lemma that

$$v(\zeta) = \left[\int_B U(z) p(z, \zeta) ds\right]\left[\int_B p(z, \zeta) ds\right]^{-1}$$

assumes the boundary value $U(z_0)$ as $\zeta \to z_0$, for a suitable singularity function $p(z, \zeta)$ such that $\int_B p(z, \zeta) ds \to \infty$ as ζ approaches any point of the boundary ($z \in B$).

The fourth method depends upon estimating

$$\int_B \left(\frac{\partial g}{\partial n}\right)^2 ds.$$

The author presents a new estimate, and points out that alternatively one can utilize a result of Payne and Weinberger [J. Math. Phys. 33 (1955), 291–307; MR 16, 923]. This fourth method extends also to the equation $\Delta u = P(x, y) u$ for $P(x, y) > 0$. *R. B. Davis* (Syracuse, N.Y.).

Misztal, F. Method of approximate solution of partial differential equations. Zastos. Mat. 2 (1956), 416–425. (Polish. Russian and English summaries)

Manfredi, Bianca. Calcolo numerico della temperatura in uno strato piano, con date condizioni al contorno variabili col tempo. Riv. Mat. Univ. Parma 6 (1955), 363–374.

The solution $u = u(x, t)$ of the boundary value problem

$$(*)\qquad \begin{aligned} &u_t = u_{xx},\ \text{in } 0 < x,\ t < 1,\\ &u(x, 0) = 0,\ 0 < x < 1;\ u(1, t) = 0,\ 0 \leqq t < 1,\\ &u(0, t) = \varphi(t),\ 0 \leqq t < 1, \end{aligned}$$

can be represented in terms of $\varphi(t)$ and a solution $v = v(x, t)$ of this boundary value problem in the special case $\varphi(t) = 1$. Based on this observation the author sets up a finite difference representation of system (*) and expresses its solution $u^{(M)}$ in terms of $\varphi(t)$ and a solution $v^{(M)}$ of the finite difference system in which $\varphi(t) = 1$. Here $M (= 1/\Delta x)$ is an integer. For the mesh ratio $0 < \Delta t/(\Delta x)^2 \leqq \frac{1}{2}$ the functions $v^{(M)}$ are shown to converge to v at the mesh points as $M \to \infty$. Several different representations of $v^{(M)}$ are given and the method is applied in the two special cases $\varphi(t) = t$ and $\varphi(t) = \sin t$.
F. G. Dressel.

Morel, Henri. Evaluation de l'erreur sur un pas dans la méthode de Runge-Kutta. C. R. Acad. Sci. Paris 243 (1956), 1999–2002.

This note gives, without details of derivation, formulas for estimating the error in using Runge-Kutta type numerical methods for estimating the solution of $y' = f(x, y)$. The error estimate is made to depend on quantities evaluated in applying the method. The formulas are given for two types of methods, one depending on three subincrements to obtain Δy, the others for four subincrements. Examples are provided. *P. C. Hammer.*

See also: Dantzig and Hoffman, p. 555; Hersch, p. 579; Harazov, p. 587; Athen, p. 603; Park, p. 612; Bishop, p. 614; Crank, p. 616; Kondo, p. 620; Przybylski, p. 628.

Graphical Methods, Nomography

Athen, H. Ein neues Verfahren zur graphischen Auswertung ganzer rationaler Funktionen. Math. Naturwiss. Unterricht 9 (1956/57), 297–301.

The new method for graphical evaluation of polynomials with real coefficients is modeled on that of E. Lill [Nouvelles Ann. Math. (2) 6 (1867), 359–362; 7 (1868), 363–367], a modern development of which is given by B. Meulenbeld [Simon Stevin 28 (1951), 60–80; MR 12, 751]. The novelty consists in using the reciprocals of the coefficients in constructing the "orthogon", which simplifies sign considerations. Although intended only as an introductory article, applications to the evaluation of polynomial derivatives and to the solution of ordinary differential equations are also given. *J. Riordan.*

Adams, Ernst. Beitrag zum Problem der schnellsten Flugverbindung zwischen zwei Punkten. Z. Flugwiss. 5 (1957), 12–15.

Für den Flug bei konstanter Höhe in einer Ebene liegt eine praktisch brauchbare graphische Approximationsmethode für den Flugweg kürzester Flugzeit zwischen zwei beliebigen Punkten bei beliebiger Windverteilung vor. Der erste Teil des Aufsatzes liefert den Nachweis, dasz diese Näherungskonstruktion unter denselben Bedingungen auch für Kugeloberflächen gilt. Der zweite Teil zeigt, wie unter Verwendung eines nur von der geographischen Breite abhängigen Längenmaszstabes die sphärischen Minimalen in einer ebenen Mercatorkarte konstruiert werden können.
Zusammenfassung des Autors.

Tables

Dingle, R. B.; Arndt, Doreen; and Roy, S. K. The integrals

$$\mathfrak{A}_p(x) = (p!)^{-1} \int_0^\infty \varepsilon^p (\varepsilon + x)^{-1} e^{-\varepsilon} d\varepsilon$$

and

$$\mathfrak{B}_p(x) = (p!)^{-1} \int_0^\infty \varepsilon^p (\varepsilon + x)^{-2} e^{-\varepsilon} d\varepsilon$$

and their tabulation. Appl. Sci. Res. B. 6 (1956), 144–154.

Dingle, R. B.; Arndt, Doreen; and Roy, S. K. The integrals

$$\mathfrak{C}_p(x) = (p!)^{-1} \int_0^\infty \varepsilon^p (\varepsilon^2 + x^2)^{-1} e^{-\varepsilon} d\varepsilon$$

and

$$\mathfrak{D}_p(x) = (p!)^{-1} \int_0^\infty \varepsilon^p (\varepsilon^2 + x^2)^{-2} e^{-\varepsilon} d\varepsilon$$

and their tabulation. Appl. Sci. Res. B. 6 (1956), 155–164.

The integrals of the titles (which arise in the theory of

semi-conductors) are tabulated, mostly to 4S, as follows: $\mathfrak{A}_p(x)$ for $p=-0.5(.5)4$ and $\mathfrak{B}_p(x)$ for $p=0(.5)4$, both for $x=0(.1)1(.2)7(.5)10(1)20$; $\mathfrak{C}_p(x)$ for $p=-0.5(.5)5$ and $\mathfrak{D}_p(x)$ for $p=0(.5)6.5$, both for $x=0(.2)2(.5)10(1)20$. It is stated that tables of $\mathfrak{A}_p(x)$ for negative x will be published later. In addition there is a table of the Fresnel integrals,

$$C(x)=\tfrac{1}{2}\int_0^x J_{-1/2}(t)dt \text{ and } S(x)=\tfrac{1}{2}\int_0^x J_{1/2}(t)dt$$

to 12D for $x=0(1)20$, prepared using their representations as the sums of Bessel functions of half-integral order, which have been tabulated by the British Association [Math. Tables Committee Report, 1925].

Recurrence relations, differential equations and asymptotic expansions of the integrals are given; in addition, relations to other special functions are pointed out. It is noted that $\mathfrak{A}_p(ix)=(p+1)\mathfrak{C}_{p+1}(x)-ix\mathfrak{C}_p(x)$, with a similar relation for $\mathfrak{B}_p(ix)$.

It is not quite clear how the tables were computed and checked. Apparently, recurrence relations, such as $p\mathfrak{A}_p(x)+x\mathfrak{A}_{p-1}(x)=1$, were used to tabulate the $\mathfrak{A}_p$, using the values of $\mathfrak{A}_{-\frac{1}{2}}(x)=(2/x)^{-\frac{1}{2}}F[(2x)^{\frac{1}{2}}]$, where $F(t)$ is the ratio of the tail area of the normal curve to its bounding ordinate at t and $\mathfrak{A}_0(x)=-e^xEi(-x)$: adequate tables of Ei and F are available. From the values of $\mathfrak{A}$ those of $\mathfrak{B}$ can be obtained from

$$\mathfrak{B}_p(x)=x^{-1}\{1-(p+x)\mathfrak{A}_p(x)\}.$$

Similarly the values of $\mathfrak{C}_p(x)$ could be obtained from expressions for $\mathfrak{C}_0(x)$, $\mathfrak{C}_1(x)$ in terms of

$$\mathrm{si}(x)=\int_\infty^x t^{-1}\sin t\,dt,\ \mathrm{Ci}(x)=\int_\infty^x t^{-1}\cos t\,dt$$

and those of $\mathfrak{C}_{+\frac{1}{2}}(x)$ in terms of $C(x)$, $S(x)$ by use of the recurrence relation $p(p-1)\mathfrak{C}_p(x)+x^2\mathfrak{C}_{p-2}(x)=1$. The values of $\mathfrak{D}_p(x)$ can then be obtained using

$$D_p=\tfrac{1}{2}x^{-2}\{(p+1)\mathfrak{C}_{p+1}(x)-(p-1)\mathfrak{C}_p(x)\}.$$

The necessity for care in the use of the recurrence relations is pointed out.

Interpolation with respect to the subscript p is encouraged, and auxiliary function for this for small x are given. Interpolation with respect to x is not discussed.
John. Todd (Washington, D.C.).

See also: Mouette, p. 561.

Machines and Modelling

Riesel, Hans. A note on large linear systems. Math. Tables Aids Comput. **10** (1956), 226–227.

Using the Gaussian elimination method, with a fixed binary point, systems of 214 equations, in which about 40% of the coefficients were different from zero, were solved on BESK in about two hours. After calculating residuals and re-solving, solutions accurate to about 10^{-6}% were obtained, even though the system was comparatively ill-conditioned. As a measure of condition, $(\Pi\,\Sigma\,|a_{ij}|)/|\det(a_{ij})|$ is suggested. *John. Todd.*

Marx, Helmut. Additionsverfahren zur Berechnung des Logarithmus und der Exponentialfunktion. I. Mitt. Math. Sem. Giessen no. **54** (1956), i+26 pp.

The author discusses, in an arithmetic based on the binary representation

$$n=\sum \varepsilon_i 2^{\alpha_i},$$

where $\varepsilon_i^2=1$ and where the α's are integers differing by 2 or more, the programming of the exponential and logarithmic functions of a complex variable without the use of multiplication or division. The method depends on a stored table of key values of the logarithms of numbers of the form $1+\varepsilon 2^{-k}$. For example $\exp(x+iy)$ can be computed to nearly 10 decimal places in at most 131 additions. A second part on the other elementary functions is promised. *D. H. Lehmer* (Berkeley, Calif.).

Bibliography of literature on questions of mathematical simulation (on analogue computing machines) (1947-1954). Avtomat. i Telemeh. **17** (1956), 279–288. (Russian)

The bibliography is organized as follows: Books; Proceedings of conferences; General theoretical questions; Electronic analogue computers; Components of electronic computers; Electromechanical analogue computers; Specialized analogue computers; Machines transferring a numerical code to a physical quantity and conversely; Application of analogue computers; Sources scanned. A continuation is promised.

See also: Hall, p. 560; Preuss, p. 626.

PROBABILITY

★ **Nolfi, Padrot. Idee und Wahrscheinlichkeit.** Editions du Griffon, Neuchatel-Suisse, 1956. 215 pp. 15 francs suisses.

This book contains an epistemological discussion of the nature of scientific theories in general with particular applications to probability. The author reviews briefly some of the conceptions of the nature of probability. Considerable space is devoted to the so-called "Spielraum" theory of von Kries, von Mises' objections to the classical probability are also refuted at some length. The author seems to feel that no all-inclusive theory of probability covering all fields of applications can be formulated. *O. Ore* (New Haven, Conn.).

Girault, M. Analycité et periodicité des fonctions caractéristiques. Publ. Inst. Statist. Univ. Paris **5** (1956), 91–94.

The author discusses some known properties of analytic characteristic functions and constructs then an interesting example of a characteristic function which is doubly periodic. This is the elliptic function

$$f(z)=\prod\frac{1-k^n}{1+k^n}\cdot\frac{1+k^ne^{iz}}{1-k^ne^{iz}},$$

where the infinite product is taken over all positive and negative odd integers n. The function has the periods 2π and $4i\log k$ and has the strip $|\mathrm{Im}(z)|<\log k$ as its strip of regularity. *E. Lukacs* (Washington, D.C.).

★ **Kuznetsov, P. I.; Stratonovich, R. L.; and Tikhonov, V. I. On the duration of excursions of random functions.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 14 pp.

Translated from Ž. Teh. Fiz. **24** (1954), no. 1, 102–112. The original Russian article was reviewed in MR **16**, 269.

★ **Fisz, Marek. Die Grenzverteilungen der Multinomialverteilung.** Bericht über die Tagung Wahrscheinlichkeitsrechnung und mathematische Statistik in Berlin, Oktober, 1954, pp. 51–53. Deutscher Verlag der Wissenschaften, Berlin, 1956.

Description of results which were given in detail elsewhere [Studia Math. **14** (1954), 272–275, 310–313; MR **16**, 839]. *J. L. Doob* (Geneva).

Blackman, Jerome. An extension of the Kolmogorov distribution. Ann. Math. Statist. **27** (1956), 513–520.

As pointed out by J. H. B. Kemperman, the results of this paper are in error. The formulas following the words "In general" at top of p. 517 are incorrect. The author is publishing a correction in which the quantities in Theorems 1 and 2 are replaced by more complicated ones and Corollaries 1 and 2 are scrapped. The salvaged formulas, if correct, no longer yield any asymptotic results.

K. L. Chung (Chicago, Ill.).

★ **Špaček, Antonin. Die Regularitätseigenschaften zufälliger Transformationen.** Bericht über die Tagung Wahrscheinlichkeitsrechnung und mathematische Statistik in Berlin, Oktober, 1954, pp. 109–111. Deutscher Verlag der Wissenschaften, Berlin, 1956.

Announcement of results which have since appeared [Czechoslovak Math. J. **5(80)** (1955), 143–151; MR **17**, 500]. *J. L. Doob* (Geneva).

★ **Mihoc, George. Über verschiedene Ausdehnungen des Poissonschen Gesetzes auf endliche konstante Markoffsche Ketten.** Bericht über die Tagung Wahrscheinlichkeitsrechnung und mathematische Statistik in Berlin, Oktober, 1954, pp. 43–49. Deutscher Verlag der Wissenschaften, Berlin, 1956.

This work begins with a presentation in the whole of the extensions given by the author to the Poisson law on the sequence of variables constituing a Markoff chain. It includes also an extension on iterations of a Markoff chain in the case of a very slow evolution.

Let $F_k(n, t)$ be the characteristic function of the iteration-number in n successive trials of a constant Markoff chain on the values $a_1, a_2, \cdots, a_m$ with the transition probabilities p_{jk} providing that $x_1=a_k$.

Thus, the following recurrence law is valid:

$$F_h(n, t)=p_{hh}F_h(n-1, t)+\sum_{k\neq h} p_{hk}e^{it}F_k(n-1, t).$$

If we are assuming that $p_{11}=1-a_1\tau$, $p_{1k}=a_k\tau$ $(a_1=a_2+\cdots+a_m)$ for p_{ik}, $i\neq1$, independent of τ, if we further are supposing, that the characteristic equation of the former recurrence law has a single root, which approaches 1, for $t\to0$, we obtain an asymptotic law of the form

$$\lim_{n\to\infty,\, n\tau\to a,\, \tau\to0} F_h(n, t)=\frac{G_h(t)}{G_1(0)}\exp\left(\frac{a\sum_2^m a_k e^{it}G_k(t)-G_1'(t)}{G_1'(t)}\right)$$

which gives us the extension of the Poisson law on iteration.

The author gives a generalization of these results also in the case, in which the characteristic equation has several roots equal to 1.

The article ends with a very simple example about a Markoff chain on two values. *O. Onicescu.*

Keilson, Julian. A suggested modification of noise theory. Quart. Appl. Math. **12** (1954), 71–76.

A study is made of Markov processes in continuous time with transition probability

$$[\tau_0(1-\gamma)]^{-1}(\beta/\pi)^{\frac{1}{2}}\exp[-\beta(x'-\gamma x)^2],$$

giving the probability density per unit time that the process if at x will jump to x'. In particular it is shown that the expected number of crossings of $x=0$ per unit time is given by

$$\frac{1}{\pi\tau_0(1-\gamma)}\tan^{-1}\left\{\left(\frac{1-\gamma}{1+\gamma}\right)^{\frac{1}{2}}\right\}.$$

As $\gamma\to1$ the process tends to the Fokker-Planck process with an infinite number of crossings. *D. V. Lindley.*

Fisz, M.; and Urbanik, K. Analytical characterization of a composed, non-homogeneous Poisson process. Studia Math. **15** (1956), 328–336.

Let $\{\xi_t,\ 0\leqq t\leqq T\}$ be a stochastic process with independent increments. It is proved that, if

$$\lim_{b-a\to0} P\{\xi_b=\xi_a\}=1.$$

then the process is a composite Poisson process, and that the characteristic function of $\xi_a-\xi_b$ is given by

$$\int_{x\neq0}(e^{itx}-1)d_x\int_a^b Q(x, I),$$

where the second integral is the Burkill integral of Q and, if $I=[a, b]$, $Q(x, I)=P[\xi_b-\xi_a<x]$ for $x<0$, $Q(x, I)=-P\{\xi_b-\xi_a\geqq x\}$ for $x>0$. The result can be considered as a consequence of Lévy's study of processes with independent increments [Théorie de l'addition des variables aléatoires, Gauthier-Villars, Paris, 1937].

J. L. Doob (Geneva).

Latscha, Robert. Zur Anwendung der kollektiven Risikotheorie in der schweizerischen obligatorischen Unfallversicherung. Mitt. Verein. Schweiz. Versich.-Math. **56** (1956), 275–302.

Phillips, D. C.; Rogers, D.; and Wilson, A. J. C. Reliability index for centrosymmetric and non-centrosymmetric structures. Acta Cryst. **3** (1950), 399.

The reliability index R when the crystal structure is nearly correct is higher for centrosymmetric structures than for non-centrosymmetric structure for the following reasons: 1) The dispersion of the centric distribution (corresponding to the centrosymmetric structure) is greater than that of the acentric distribution (corresponding to the non-centrosymmetric structure); 2) the average value of $|F|$ is less for the centric distribution; 3) the proportion of extremely weak intensities is larger for the centric distribution. The relative effects on R of aech of the latter two factors are computed.

H. A. Hauptman (Washington, D.C.).

Wilson, A. J. C. Largest likely values for the reliability index. Acta Cryst. **3** (1950), 397–398.

The reliability index R, defined by

$$R=\frac{\sum\left||F_{\text{obs}}|-|F_{\text{calc}}|\right|}{\sum|F_{\text{obs}}|},$$

where F is the structure amplitude, is commonly used as a measure of the quality of a crystal structure determination. The probable value of R for a completely wrong structure is here found to be 0.828 for a centrosymmetric structure

and 0.586 for a non-centrosymmetric structure.
H. A. Hauptman (Washington, D.C.).

See also: Henstock, p. 584; Bottema, p. 629.

STATISTICS

Pearson, E. S. Some aspects of the geometry of statistics. The use of visual presentation in understanding the theory and application of methametical statistics. J. Roy. Statist. Soc. Ser. A. **119** (1956), 125–146.

This paper is the Inaugural Address of its author as President of the Royal Statistical Society. It is a partly historical and partly expository presentation of geometrical representation in statistics. In the historical section, the author gives a brief account of Karl Pearson's use of geometry in statistics. Most of the paper is devoted to geometrical diagrams for various problems in statistics such as testing the difference between means, student's test, components in the analysis of variance, analysis of covariance, and preliminary examination of a set of data as a whole for patterns. *S. S. Wilks* (Princeton, N.J.).

Benedetti, Carlo. Sulla rappresentabilità di una distribuzione binomiale mediante una distribuzione B e viceversa. Metron 18 (1956), no. 1-2, 121–131.

The author replaced the binomial discontinuous distribution

$$f_b(x)=\binom{n}{nx}p^{nx}(1-p)^{n-nx},$$

where $x=0, 1/2, \cdots, (n-1)/n$, 1 by a continuous distribution of type B,

$$f_B(x)=\frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m)}x^{l-1}(x-1)^{m-1}.$$

The parameters l and m are determined by the conditions of equality between the mean and the variance of the two distributions. The latter distribution corresponds, following the conceptions of Gini on this matter, to a uniform distribution of the causes and is much easier to handle in the applications than the original binomial distribution. *O. Onicescu* (Bucarest).

Harkin, B. The expected error of a least-squares solution of location from direction-finding equipment. Austral. J. Appl. Sci. 7 (1956), 263–272.

Several stations simultaneously report the bearing and elevation from each of a missile being tracked by them, and the missile is taken as located at that point whose squared distances in space from the generally skew lines sum to a minimum. The author deduces the error variances and covariances of the coordinates. The formulas are too long to be given here, but they and their derivation could be vastly reduced by the use of matrices and vectors. *A. S. Householder* (Madison, Wis.).

Robbins, Herbert. Sequential decision problem with a finite memory. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 920–923.

A decision rule is proposed which may be the solution of the following problem. Given two coins with unknown probabilities of heads, p_1, p_2, and given an integer r, to choose at the nth step one of the coins to toss, as a function of the results of the last r tosses, so as to maximize the long-run proportion of heads. The rule proposed is to change coins when (a) one coin gives r successive tails, or (b) a coin gives tails on the first toss after changing from the other coin. It is shown that the rule does well [better than if (b) were omitted], and as r increases the limiting frequency converges [uniformly in p_1, p_2] to max (p_1, p_2). {Reviewer's remark: For $r=2$ there are 256 symmetric rules, mostly foolish ones; the author's rule appears to be optimal in this case.} *J. Isbell* (Princeton, N.J.).

Freund, John E. Some methods of estimating the parameters of discrete heterogeneous populations. J. Roy. Statist. Soc. Ser. B. **18** (1956), 222–226.

The problem is to estimate the prior distribution $f_j=f(\theta_j)$ of a parameter θ in a heterogeneous population $(j=1, 2, \cdots, \alpha)$, where the conditional distributions $f_{ij}=f(x_i|\theta_j)$ of the variable x are known $(i=1, 2, \cdots, \beta)$, α and β being finite. The value x_i is observed with frequency n_i in a random sample of size n. It is shown that for $\beta<\alpha$ there are no linear unbiased estimates (l.u.e.) of the f_j; for $\beta=\alpha$, and if the rank r of the matrix of f_{ij} is equal to α, the maximum likelihood estimates are the unique l.u.e.'s; for $\beta>\alpha$ there is a class of l.u.e.'s, from which it is proposed to select a unique set f_j' by a minimax procedure: this minimises the maximum (with respect to the f_j) of a convenient upper bound to the sampling variance $\sum_j E(f_j'-f_j)^2$. An example is given.
P. Armitage.

Des Raj. On the method of overlapping maps in sample surveys. Sankhyā 17 (1956), 89–98.

The optimum sampling schemes for estimating two distinct characteristics of a population (e.g., area and number of residents for a collection of villages) may assign different probabilities to the sampling units. The problem of minimizing costs in such a sampling situation is shown to be equivalent to the "transportation problem" [Dantzig, Activity analysis of production and allocation, Wiley, New York, 1951, pp. 359–373; MR 15, 48].

Dantzig's "basic solution", used as a first approximation in the simplex method, is shown to be optimum for a certain definition of "distance" between sampling units. For other situations an approximation is given which usually leads to the optimum more quickly.

It is noted that two similar problems in sampling theory [Goodman and Kish, J. Amer. Statist. Assoc. **45** (1950), 350–372; Keyfitz, ibid. **46** (1951), 105–109] are equivalent to this one. *P. Meier* (Baltimore, Md.).

Gumbel, E. J.; and Carlson, P. G. On the asymptotic covariance of the sample mean and standard deviation. Metron 18 (1956), no. 1-2, 113–119.

Let $\bar{x}$ and $s=\sqrt{m_2}$ be the mean and standard deviation of a sample of size n from a population for which all moments exist. Let μ_i be the ith population central moment, $i=1, 2, \cdots$. The main result of this paper is to show the asymptotic formula

$$\text{Covariance }(\bar{x}, s)=\mu_3(2n\sqrt{\mu_2})+O(n^{-3/2})$$

and hence for any assigned pair of constants a and b

$$\text{Var.}(a\bar{x}+bs)=a^2\frac{\mu_2}{n}+b^2\frac{\mu_4+\mu_2^2}{4n\mu_2}+O(n^{-3/2}).$$

The applications to specific distributions such as the normal, the logistic and limited exponential distributions are given. The results of this paper lead to the errors of

estimate for two-parameter distributions, where the parameters are estimated by linear functions of the mean and standard deviations. *T. Kitagawa* (Fukuoka).

Reiter, Stanley. Estimates of bounded relative error for the ratio of variances of normal distributions. J. Amer. Statist. Assoc. **51** (1956), 481–488.

For estimates for $\tau=\sigma_1^2/\sigma_2^2$, the paper considers estimates of the form $\hat\tau=\beta\hat\sigma_1^2/\hat\sigma_2^2$ where $n_1\hat\sigma_1^2/\sigma_1^2$ and $n_2\hat\sigma_2^2/\sigma_2^2$ are assumed to be independently distributed like chi-square with n_1 and n_2 degrees of freedom, respectively. The condition that $\hat\tau$ be of bounded relative error with confidence can be represented by

$$P\left\{\frac{1}{K}\leq\frac{\hat\tau}{\tau}\leq K\right\}=\alpha(K)\quad(K>0),$$

where $\alpha(K)$ does not depend upon τ and hence can be denoted by $\psi_K(b, n_1, n_2)$. The paper shows that the maximum of this value, for fixed K, n_1 and n_2, is given by

$$b^*=\frac{K^{1/(1+\theta)}-K^{-1/(1+\theta)}}{K^{\theta/(1+\theta)}-K^{-1/(1+\theta)}},$$

where $\theta=n_1/n_2$, for which a numerical table is given in the paper. *T. Kitagawa* (Fukuoka).

Bennett, B. M. On the use of preliminary tests in certain statistical procedures. Ann. Inst. Statist. Math., Tokyo **8** (1956), 45–52.

The problems treated in this paper start with the exact formulations of common statistical procedures on the poolings of data, and belong to the realm of successive process of statistical inferences. This paper deals with the use of preliminary tests in providing interval estimates of the mean and variance of normal populations. The author proceeds to discuss a procedure in normal regression theory involving a decision about the appropriate degree of the orthogonal polynomial. There remain numerical calculations of various integrals obtained by the author in order to make practical use of these procedures.

T. Kitagawa (Fukuoka).

Basu, D. The concept of asymptotic efficiency. Sankhyā **17** (1956), 193–196.

This paper introduces a "concentration" partial ordering on estimators of the value of a real function μ on a class Ω of distributions F: $\{t_n\}\geq\{t_n'\}$ if and only if

$$\limsup_{n\to\infty}\frac{\Pr[|t_n-\mu|>\varepsilon|F]}{\Pr[|t_n'-\mu|>\varepsilon|F]}\leq 1 \text{ for all } F\in\Omega,\ \varepsilon>0.$$

For asymptotically $-N(\mu(F), \sigma_n(F))$-estimators, $\geq$ is shown to be incomparable to the classical ordering induced by $\limsup \sigma_n(F)/\sigma_n'(F)$. *J. Hannan.*

Ruben, H. On the moments of the range and product moments of extreme order statistics in normal samples. Biometrika **43** (1956), 458–460.

The author shows that the product moments of the extreme order statistics in samples of even sizes from normal populations can be expressed as linear functions of the products of the contents of certain hyperspherical simplices. Using this result the author obtains simple explicit expressions for the variance of the sample range for samples of sizes 2 and 4. *D. M. Sandelius* (Göteborg).

Billingsley, Patrick. Asymptotic distributions of two goodness of fit criteria. Ann. Math. Statist. **27** (1956), 1123–1129.

We are given a sequence of n digits, each in the 'alphabet' 1, 2, $\cdots$, s. Let $\mathbf{u}=(u_1, u_2, \cdots, u_\nu)$ be any ν-plet of digits and let $n_{\mathbf{u}}$ be its frequency in the given sequence. Let H_μ' be the composite hypothesis of Markovity of order μ ($\mu=-1, 0, 1, 2, \cdots$; where by convention H_{-1} is the hypothesis of equiprobable or perfect randomness). Let H_μ be any simple statistical hypothesis belonging to H_μ'. Let $\hat H_\mu$ be the maximum-likelihood H_μ. Let the expected value of $n_{\mathbf{u}}$ in a new sequence of length n from the same ensemble given H_μ be $n_{\mathbf{u},\mu}$ and given $\hat H_\mu$ be $\hat n_{\mathbf{u},\mu}$. Let

$$\psi^2_{\nu,\mu}=\sum_{\mathbf{u}}\frac{(n_{\mathbf{u}}-n_{\mathbf{u},\mu})^2}{n_{\mathbf{u},\mu}},\quad \hat\psi^2_{\nu,\mu}=\sum_{\mathbf{u}}\frac{(n_{\mathbf{u}}-n_{\mathbf{u},\mu})^2}{\hat n_{\mathbf{u},\mu}},\quad \hat\psi^2_{\mu+1,\mu}=0.$$

It may be conjectured [cf. Good, Biometrika **42** (1955), 531–533, sect. 3; this paper contains some errors privately communicated by L. A. Goodman; MR **17**, 381] that when H_μ is true, $\hat\psi_{\nu,\mu}{}^2$ has asymptotically (when $n\to\infty$) a distribution

$$\mathop{*}_{\lambda=1}^{\nu-\mu-1} K_{s^{\nu-1-\lambda}(s-1)^2}(x/\lambda),$$

where $*$ denotes convolution, and where $K_i(x)$ is the gamma-variate distribution with i degrees of freedom. This conjecture is proved by Billingsley for the case $\mu=0$. It may further be conjectured that when H_μ is true, $\psi_{\nu,\mu}{}^2$ has asymptotically the distribution

$$\mathop{*}_{\lambda=1}^{\nu-1} K_{s^{\nu-1-\lambda}(s-1)^2}(x/\lambda)*K_{s-1}(x/\nu)\quad\text{(math. independent of } \mu).$$

This was proved by Good [Proc. Cambridge Philos. Soc. **49** (1953), 276–284; MR **15**, 727] for the case s prime and $\mu=-1$. It is now proved for the case $\mu=0$ by using the theory of finite-dimensional vector spaces. An acknowledgement is given to R. H. Spanier for some of the essential algebraic ideas. *I. J. Good* (Cheltenham).

David, F. N. A note on Wilcoxon's and allied tests. Biometrika **43** (1956), 485–488.

Given two independent samples let $x_1, \cdots, x_{n_1}$ be the ranks of one of the samples in the sequence formed by combining both samples. As tests for hypotheses of equal location and spread of the corresponding distributions (assumed identical if these hypotheses are true) the author suggests

$$(*)\quad \{\bar x_1-E(\bar x_1)\}/\{\operatorname{var}(\bar x_1)\}^{-\frac12}\ \text{and}\ (**)\quad s_1^2/E(s_1^2),$$

where $\bar x_1=\sum_{i=1}^{n_1}x_i/n_1$ and $s_1^2=\sum_{i=1}^{n_1}(x_i-\bar x_1)^2/(n_1-1)$. For sufficiently large sizes of both samples (*) is considered approximately normally distributed, while for (**) a Pearson curve approximation is used. *D. M. Sandelius.*

Gurland, John. Quadratic forms in normally distributed random variables. Sankhyā **17** (1956), 37–50.

Let $X_1, X_2, \cdots, X_n$ be independently and normally distributed with zero mean and variance one.

The author rewrites the characteristic function $\phi(t)=\prod_{j=1}^n(1-2i\lambda_jt)^{-\frac12}$ of $\sum_{i=1}^n \lambda_iX_i^2$ in the form $\phi(t)=(1-2i\lambda t)^{-n/2}\prod_{j=1}^n(1-2i\alpha_jt)/(1-2i\lambda t))^{-\frac12}$ with $\alpha_j=\lambda_j-\lambda$ and $\lambda>\frac12\max\lambda_i$, which, in combination with the inversion formula, gives the convergent series expansion of the distribution function $F(x)$ of $\sum_{i=1}^n\lambda_iX_i^2$, and the evalua-

tion formula for the remainder of partial sums of the series. The paper then discusses the similar problems associated with the statistics

$$\sum_{i=1}^{n_1} \lambda_i X_i^2 - \sum_{i=n_1+1}^{n} \lambda_i X_i^2$$

and with the ratio of quadratic forms and gives a bound on their remainder terms in partial sums of convergent expansion. The method is a continuation of the one adopted by the author in several of his papers [Ann. Math. Statist. **19** (1948), 228–237; **24** (1953), 416–427; **26** (1955), 122–127; MR **9**, 582; **15**, 885; **16**, 727] and has an intimate connection with that of Bhattacharya [Sankhyā **7** (1945), 27–28; MR **7**, 131]. *T. Kitagawa.*

Fisher, Ronald. On a test of significance in Pearson's Biometrika Tables (No. 11). J. Roy. Statist. Soc. Ser. B. **18** (1956), 56–60.

The author derives the distribution of

$$d=(x_1-\bar{x}_2)(s_1^2+s_2^2)^{-1/2}$$

under the assumption of normality given

$$s_1^2=\sum (x_1-\bar{x}_1)^2/n_1(n_1+1),\ s_2^2=\sum (x_2-\bar{x}_2)^2/n_2(n_2+1),$$
$$\bar{x}_1=\sum x_1/n_1+1,\ \bar{x}_2=\sum x_2/n_2+1,$$

n_1 and n_2 the degrees of freedom. This result depends on the unknown population variances $\sigma_1^2(n_1+1)$ and $\sigma_2^2(n_2+1)$. In the special case $n_1=n_2$, $s_1=s_2$, $d/\cosh z$ is distributed as Student's t with n_1+n_2 degrees of freedom where z is defined by $\sigma_2/\sigma_1=s_2/s_1e^z$. For $n_1=n_2=6$, using the 10 per cent critical value of d from table 11 of "Biometrika tables for statisticians", v. 1 [Cambridge, 1954; MR **16**, 53] the author shows that for $z=0$, the level of significance is about 11 per cent, and for all values of z the level of significance is always greater than 10 per cent. The theory on which the table is based is due to Welch [Biometrika **34** (1947), 28–35; MR **8**, 394]. The point of view of the author has been expressed, for instance, by Fisher [Ann. Eugenics **9** (1939), 174–180]. This problem has also been considered by Wald [Selected papers in statistics and probability, McGraw-Hill, New York, 1955, pp. 669–695; MR **17**, 55]. *L. A. Aroian.*

Rao, C. Radhakrishna. Analysis of dispersion with incomplete observations on one of the characters. J. Roy. Statist. Soc. Ser. B. **18** (1956), 259–264.

This paper deals with the multivariate generalization of the analysis of variance (dispersion) in which the Λ criterion for testing hypotheses on p multiple characters simultaneously is not applicable because the data are incomplete for some one character. The author suggests the criterion $\Lambda=\Lambda_1\Lambda_2^{(n-c)/n}$, where Λ_1 is calculated for the $p-1$ characters on which the information is complete for a sample of $n+1$ individuals, and where Λ_2 is calculated for the $n+1-c$ individuals for which the information is complete on all characters. An asymptotic expansion of the distribution of the test criterion is developed in a χ^2 series, for which the first term provides a good approximation for moderately large n. A second method which compares the tth moment of Λ with that of a β distribution gives a less satisfactory result. A numerical problem illustrates the methods. *P. S. Dwyer.*

Bozivich, Helen; Bancroft, T. A.; and Hartley, H. O. Power of analysis of variance test procedures for certain incompletely specified models. I. Ann. Math. Statist. **27** (1956), 1017–1043.

Let the sums of squares n_iV_i be independently distributed as $\chi_i^2\sigma_i^2$, where χ_i^2 is the central χ^2 statistic based on n_i degrees of freedom. The paper is interested in testing the hypothesis H_0: $\sigma_3^2=\sigma_2^2$, against the alternative H_1: $\sigma_3^2>\sigma_2^2$, when it is known that $\sigma_3^2\geqq\sigma_2^2\geqq\sigma_1^2$. The test procedure with some times-pooling V_2 and V_1 is then as follows: Reject H_0 if either

$$\{V_2/V_1\geqq F_{n_2n_1}(\alpha_1) \text{ and } V_3/V_2\geqq F_{n_3n_2}(\alpha_2)\}$$

or

$$\{V_2/V_1\leqq F_{n_2n_1}(\alpha_1) \text{ and } V_3/V\geqq F_{n_3n_1+n_2}(\alpha)\},$$

where $V=(n_1V_1+n_2V_2)/(n_1+n_2)$ and $F_{n_i,n_j}(\alpha)$ is the upper $100\alpha\%$ point of the F-distribution with numerator $df=n_i$ and denominator $df=n_j$.

This procedure is called the sometimes-pool procedure, and has been advocated under the wide application of analysis of variance to operational research and to the study of routine data in which our model is incompletely specified. The procedure has been discussed by Bancroft, Mosteller, Paull, Kitagawa, Bechhofer, Bennett and so on. The object of this paper is to provide the necessary extension of Paull's investigation which was made possible by (i) the development of the power integrals as series formulas for even values of the degrees of freedom n_1, n_2 and n_3; (ii) the derivation of recurrence formulas for the power for even values of n_1, n_2, and n_3; and (iii) the development of approximate formulas valid for large degrees of freedom for even values of n_1, n_2 n_3.

The paper gives some practical recommendations, considering the relative merits of procedures at $\alpha_1=.25$, $\alpha_1=.50$ and $\alpha_1=.7$ to .8 (the borderline level) to the following effect. (i) If the experimenter is reasonably certain that only small values of θ_2 can be envisaged as a possibility, he is advised to use $\alpha_1=.25$ except in the cases $n_3\geqq n_2$ and $n_1\geqq 5n_2$ when he should use $\alpha_1=.50$, in order to ensure size control. (ii) If, however, the experimenter can make no such assumption about θ_2 and wishes to guard against the possibility of power losses, he may then use the borderline test, which would ensure a power gain.

The paper contains detailed analytical and numerical studies about the power and size curves and comparison of test procedures. *T. Kitagawa* (Fukuoka).

★ **Rider, Paul R.; Harter, H. Leon; and Lum, Mary D. An elementary approach to the analysis of variance.** WADC Tech. Rep. 56-50. Wright Air Devel. Center, Wright-Patterson Air Force Base, Ohio, 1956. iv+65 pp.

This report is an elementary exposition of the analysis of variance whose objective is to present this powerful statistical technique in such a manner that will be directly useful to engineers and other scientists. The report provides us with several sections most of which are now quite familar in these branches. However, it also includes some emphasis upon newly developed notions and technique such as hierarchical and partially hierarchical models, fractionally replicated experiments and multiple comparisons, which will make the use of analysis of variance more powerful, especially in engineering applications. *T. Kitagawa* (Fukuoka).

Cane, Violet R. Some statistical problems in experimental psychology. J. Roy. Statist. Soc. Ser. B. **18** (1956), 177–194; discussion 195–201.

The first part of the paper deals with the determination of perceptual thresholds. A subject is presented with standard and variable stimuli, S and V, respectively, and

perceives $S'=S+s$ and $V'=V+v$, where s and v are independent random variates. He can distinguish between S and V only when $|S'-V'|>C$. Three different methods for estimating C are compared: (a) where different values of V are presented in random order, (b) where they occur in ascending or descending order, and (c) where the order is determined by the subject. Some experimental results on visual thresholds are interpreted in terms of a particular psychophysical model. The second part of the paper concerns learning experiments, which are interpreted in terms of renewal theory.

P. Armitage (Washington, D.C.).

Bliss, C. I. Confidence limits for measuring the precision of bioassays. Biometrics **12** (1956), 491–526.

Explicit formulae are given for confidence limits for the potency ratio in various types of assay. *P. Armitage.*

Bastenaire, François. Sur une propriété des distributions statistiques des durées de vie à la fatigue. C. R. Acad. Sci. Paris **243** (1956), 1270–1273.

The limitations are discussed that are imposed on the statistical distribution functions of fatigue-life at constant stress within the stress-range in which a number of test-specimens will always survive, independently of the number of stress-cycles applied. It is pointed out that within this range the statistical expectancy and the variance of the fatigue life can not be defined, so that the mean and variance observed in a test sample are statistically meaningless. *A. M. Freudenthal.*

See also: Keeping, p. 602; Banerjee, p. 630; Li, p. 630.

PHYSICAL APPLICATIONS

Mechanics of Particles and Systems

Savin, G. N. On the fundamental equations of the dynamics of a shaft-lifting cable. Akad. Nauk Ukrain. RSR. Prikl. Meh. **1** (1955), 5–22. (Ukrainian. Russian summary)

Consider a lifting cable which hangs over a cylinder revolving about a fixed horizontal axis with velocity v. At the lower end B of the cable a load of weight Q is attached which at the beginning of the motion lies on a fixed support. Assume that the lifting of the load proceeds according to a trapezoidal tachogram with constant acceleration. The cable being an elastic viscous string, consider the dynamic pull caused in it during the lifting of the load Q. Let the dynamic pull caused at the lower end B of the cable while the load remains on the support be $T_0=\zeta Q$, where $0\leq\zeta\leq1$ characterizes the degree of slackness of the lifting cable.

Let the OX-axis be directed downward along the axis of the cable, the origin O being the point C at which the cable starts to wind onto the cylinder. Consider a section of the cable at a point A and let

$$X=\xi-x-u(x,t),$$

where x denotes the unstretched length of the segment AB of the cable, $u(x,t)$ the absolute lengthening of AB and $\xi=BC$ the variable length of the cable. As the cylinder starts to revolve the lifting cable first streches out during an interval of time τ, the load Q remaining fixed on the support (for $\zeta\neq1$). The load Q will start to move upward only after the dynamic pull at the lower end B of the cable has attained the value Q. During the first phase of the lifting process $\xi=$const, during the second phase $\xi=\xi(t)$.

The fundamental differential equations of motion for an element of the cable at A and the load Q are

$$(1)\qquad \frac{q}{g}\left(\frac{d^2\xi}{dt^2}-\frac{\partial^2u}{\partial t^2}\right)=-\frac{\partial T}{\partial x}+q,$$

$$(2)\qquad \frac{Q}{g}\frac{d^2\xi}{dt^2}=Q-T(O,t)+R,$$

where R is a resistance force and q a constant. The boundary condition at the lower end B of the cable is (3), $u(0,t)=0$ and at the upper end C is $(\partial X/\partial t)_{x=l}=v$ or

$$(4)\qquad \frac{d^2\xi}{dt^2}=\frac{dv}{dt}+\left(\frac{\partial^2u}{\partial t\partial x}\right)_{x=l}\frac{dl}{dt}+\left(\frac{\partial^2u}{\partial t^2}\right)_{x=l}$$

(l denotes the normal length of the cable at the instant t). The initial conditions of $u(x,t)$ and $\partial u/\partial t$ for the second phase of the lifting process are

$$(5)\qquad u(x,O)=m_1x+m_2x^2,\quad (\partial u/\partial t)_{t=0}=m_3x,$$

where m_1, m_2, m_3 are constants. The dynamic pull acting in the cross-section A of the cable is assumed to be of the form (using complex notations)

$$T=K\left(1+i\frac{\psi}{2\pi}\right)\frac{\partial u}{\partial x},$$

where $\psi=2\delta$, δ being the logarithmic decrement of damping, found for the given cable from the oscillogram of free vibrations of the attached load, and K is a constant.

In the case of small lifting depths $u(x,t)=x\varepsilon(t)$, and the problem reduces to the integration of an ordinary second order linear differential equation with variable coefficients for $\varepsilon(t)$.

Further, using the idea of the method of moments, the author reduces the equations (1), (2) with the boundary conditions (3), (4) and the initial conditions (5) to an integro-differential system from which certain approximate solutions can be obtained. In particular, if $u(x,t)=x\varphi(t)+x^2\phi(t)$, the problem reduces to the integration of two ordinary second order differential equations with variable coefficients for $\varphi(t)$ and $\phi(t)$ [cf. Savin, Ukrain. Mat. Ž. **6** (1954), 126–139; MR **16**, 1060].

E. Leimanis (Vancouver, B.C.).

Sokolov, Yu. D. On the determination of dynamic pull in shaft-lifting cables. Akad. Nauk Ukrain. RSR. Prikl. Meh. **1** (1955), 23–35. (Ukrainian. Russian summary)

This paper contains a detailed discussion of the basic partial differential equation

$$\frac{g}{q}\frac{\partial T}{\partial x}-v^2\frac{\partial^2u}{\partial x^2}=\frac{\partial^2u}{\partial t^2}+2v\frac{\partial^2u}{\partial x\partial t}-v\left(\frac{\partial^2u}{\partial x\partial t}\right)_{x=0}+\frac{dv}{dt}-g$$

for a lifting cable as obtained by G. N. Savin [Dopovidi Akad. Nauk Ukrain. RSR **1954**, 140–147] under the assumption that

$$T(x,t)=K\frac{\partial u}{\partial x}+\alpha\frac{\partial^2u}{\partial x\partial t},$$

where α characterizes damping of the dynamic pull in

the cable and K denotes a constant. Otherwise the paper follows along the lines outlined in a previous paper by the same author [ibid. **1955**, 21–25; MR **17**, 307].

E. Leimanis (Vancouver, B.C.).

Noskov, N. I. Construction by the method of dyads of a seven link mechanism. Trudy Inst. Mašinoved. **16** (1956), no. 62, 5–10. (Russian)

On étudie le mouvement d'un mécanisme avec 7 membres, c'est à dire avec 7 couples rotationels dont les axes $AB, CD, \cdots, PQ$ sont gauches. Les deux dyades $CDEFGH$, $GHKLMN$ qui composent le mécanisme ont les axes CD, GH, MN qui peuvent posséder les divers positions. L'auteur remplace les couples mobiles (CD, HG), (HG, MN) par les couples cylindriques et il cherche les positions nulles des axes à l'aide des méthodes graphiques de la géométrie descriptive. La méthode donne un résultat approximatif.

F. Vyčichlo (Prague).

Božilov, Boris. Transverse vibrations of a string with varying density. Trudy Vysšego Inst. Narod. Hozyaĭstva Staline. Inž.-Stroitel. Fak. **1** (1954), 67–86. (Bulgarian. Russian summary)

The author considers a string stretched along the x axis between two fixed points with the usual assumptions connected with small transverse displacements. The density of the string is a continuous function of x. He derives first a Fredholm integral equation which he solves for a special case when the density is a quadratic function of x, obtaining three modes and three characteristic frequencies. Then he transforms the Fredholm equation into a Volterra integral equation and solves it again for the same special case.

T. Leser (Aberdeen, Md.).

Bishop, R. E. D. The behaviour of damped linear systems in steady oscillation. Aero. Quart. **7** (1956), 353–354.

This is an addendum to a previous paper of the same title [Aero. Quart. **7** (1956), 156–168; MR **17**, 1246]. A more penetrating analysis of one set of equations therein enables the author to clarify more completely the reasons for observing different vibration shapes when an airplane structure is excited at different places.

E. Pinney.

Liverani, Giovanni. Su un teorema di reciprocità per i sistemi a ritardo. Boll. Un. Mat. Ital. (3) **11** (1956), 582–584.

Si consideri un sistema meccanico, a n gradi di libertà, definito delle coordinate lagrangiane $q_1, q_2, \cdots, q_n$, soggetto a forze conservative, che compia, attorno a una posizione di equilibrio stabile, piccoli movimenti forzati, prodotti da sollecitazioni addizionali di componenti lagrangiane $Q_1, Q_2, \cdots, Q_n$, assegnate funzioni del tempo t. $\cdots$Sono stati però considerati i cosidetti sistemi meccanici a „ritardo", retti da equazioni miste differenziali e alle differenze. Tali equazioni si possono ottenere$\cdots$ in principio sostituendo, in alcuni termini, alla q_i calcolata nell'instante generico t, le stesse grandezze calcolate in un istante antecedente $t-\tau$ dove τ è, di solito, una costante che esprime appunto il ritardo con cui certe azioni si manifestano.

Dal riassunto dell'autore.

Matschinski, M. Principes mathématiques de toute théorie des périodes glaçiaires. J. Sci. Météorol. **8** (1956), 69–97. (English and Spanish summaries)

The author's object is to establish a physico-mathematical theory of glaciation-periods. In order to accomplish this task, the author writes down a system of three ordinary (non-)linear differential equations. (The essential derivations are not indicated.) Thereafter, the author transforms these equations, obtains 'solutions', plots graphs, etc.

{It seems to the present reviewer that not too much significance can be attached to the author's 'solutions'. The author's analysis does not even satisfy some of the most rudimentary principles of physico-mathematics. Also, the equations postulated and the solutions presented lack completeness. Unfortunately, it is not possible to treat boundary-value problems of nonlinear equations in such a simple manner.}

K. Bhagwandin (Oslo).

See also: Fischer, p. 560; Göransson and Hansson, p. 577.

Statistical Mechanics

★ **Temperley, H. N. V. Changes of state. A mathematical-physical assessment.** Cleaver-Hume Press Ltd., London; Interscience Publishers Inc., New York, 1956. xi+324 pp. $7.50.

The commonness of phase changes in matter, such as the melting of solids, is not paralleled by a corresponding simplicity in their theory. The author has undertaken the herculean task of reviewing this field critically, with main attention to the physical and mathematical interpretations of the various models rather than to their computational details. The thermodynamic approach is reviewed and Ehrenfest's classification of orders of phase changes extended. After a discussion of the general nature of phase changes attention is put on special types, such as ferromagnetism, superconductivity, and liquid helium. It would not be appropriate to review this material topically here, and the mathematical reader would be ill advised by the suggestion that these problems can be reduced to an axiomatic basis. The underlying difficulty is to find variables suitable for the description of the macroscopic properties of complex systems of many particles. Nevertheless, some of the models which have been proposed are sufficiently clear to lead to highly intriguing mathematical problems. The mathematician who is interested in the physical applications of thermodynamics and statistical mechanics, and is willing to look up the relevant experimental results, will find here a well-balanced and helpful analysis of the field from which he can venture into the jungle of special problems.

E. L. Hill (Minneapolis, Minn.).

Yamamoto, Tsunenobu; and Matsuda, Hirotsugu. On the grand canonical distribution method of statistical mechanics. Progr. Theoret. Phys. **16** (1956), 269–286.

The main object of the paper is to establish that the thermodynamical quantities calculated for a classical system on the basis of the grand canonical theory are the same as those which are obtained for a macroscopic cell of the system by considering the latter in the microcanonical distribution. This is shown to be the case for the pressure as function of temperature and chemical potential. The equivalence of the grand canonical, canonical and microcanonical expressions of the entropy is also established. The method used consists in a subdivision of the whole system into a large number of macroscopic cells, whereby the interactions across cell walls are shown to be negligible. The author does not seem to be aware of the fact that this method was already

applied to the grand canonical theory by Yang and Lee [Phys. Rev. (2) **87** (1952), 404–409; MR **14**, 711].
L. Van Hove (Utrecht).

Ziman, J. M. The general variational principle of transport theory. Canad. J. Phys. **34** (1956), 1256–1273.

The Boltzmann integral equation governing transport phenomena in systems which are in quasi-statistical equilibrium is formulated in terms of a variational principle. This has the advantage in a mathematical sense that it allows for solution by pertubation techniques with approximating sequences. The variational principle itself is interpreted in terms of the rate of macroscopic entropy production, assuming, as usual, that entropy can be defined meaningfully in non-equilibrium states. The connection with Onsager's macroscopic relations is given, and a number of special applications are cited. The difficulties of principle which arise when an external magnetic field is introduced are discussed. *E. L. Hill.*

Kubo, R. A general expression for the conductivity tensor. Canad. J. Phys. **34** (1956), 1274–1277.

The general transport problem is formulated in terms of the perturbation produced in the statistical density function by the external field. The formal result is very compact and useful for the discussion of general relationships, and provides a parallel to the discussion by Ziman in the paper reviewed above. In explicit calculations some equivalent of the Boltzmann transport equation would be required by the necessity of giving explicit definition to the state space of the system. *E. L. Hill.*

Prigogine, I. On the statistical mechanics of irreversible processes. Canad. J. Phys. **34** (1956), 1236–1245.

This paper, which was read by the author at the International Conference on Electron Transport in Metals and Solids held at Ottawa in September 1956, describes recent developments in the theory of irreversible processes for systems, the dissipative behaviour of which is due to a small perturbation. The case of classical systems is treated. The treatment is based on the use of action and angle variables, in which the characteristic properties of the perturbation responsible for its irreversible effects are most simply expressed. These properties, first recognized and studied by the reviewer for quantum systems [Physica **21** (1955), 517–540; MR **17**, 115], were formulated and investigated by R. Brout in collaboration with the author for classical systems [ibid **22** (1956), 35–47, 263–272, 621–636]. The present paper may be considered as a very good review of the latter work. *L. Van Hove.*

Mori, Hazime. A quantum-statistical theory of transport processes. J. Phys. Soc. Japan **11** (1956), 1029–1044.

The author discusses for quantum systems the relation between the molecular properties and the phenomenological equations of transport processes. The actual motion of the system is considered for a short interval of time and the resulting expression for the time variation of the relevant physical quantities is compared with the predictions of the phenomenological theory. The conditions under which the phenomenological equations can be valid are indicated and relations are found between the phenomenological coefficients and equilibrium fluctuations. The fluctuation dissipation theorem is thereby extended to transport processes which cannot be described as resulting from an external force term in the hamiltonian. A new molecular expression of the entropy is proposed with the property that its time-variation obeys Gibbs' relation, as is required for comparison with the phenomenological theory. Simple examples are considered to illustrate the various parts of the discussion.
L. Van Hove (Utrecht).

Born, M.; and Hooton, D. J. Statistical dynamics of multiply-periodic systems. Proc. Cambridge Philos. Soc. **52** (1956), 287–300.

It is argued that even in classical mechanics the impossibility of determining the exact initial values of p, q gives rise to an uncertainty Δp, Δq. [Cf. P. Duhem, The aim and structure of physical theory, Princeton, 1954, ch. 3; MR **15**, 387.] Hence the observed state of a system should be represented by a probability density in phase space, which tends to spread in the course of time over the whole space. This process is worked out for a multiply-periodic system, consisting of a set of harmonic oscillators. [For a somewhat simpler example see R. Becker, Theorie der Wärme, Springer, Berlin-Göttingen-Heidelberg, 1955, p. 108.] *N. G. van Kampen.*

Bayet, Michel; Delcroix, Jean-Loup; et Denisse, Jean-François. Théorie cinétique des plasmas homogènes faiblement ionisés. III. L'opérateur de collision dans le cas du gaz de Lorentz imparfait. J. Phys. Radium (8) **17** (1956), 923–930.

The work of parts I and II [same J. (8) **15** (1954), 795–803; **16** (1955), 274–280; MR **16**, 550, 890] on the perfect Lorentz gas is extended to the imperfect gas by studying the effects of energy exchange between the electrons and molecules to order m/M, the ratio of the masses. It is shown that this has a negligible effect upon the anisotropic part of the velocity distribution but an important influence on the isotropic part. The collision operator for collisions between the isotropic electron distribution and a Maxwellian distribution for the molecules is analysed in detail including a study of the characteristic functions and their characteristic values for various types of force fields. Applications of these results are to be discussed in part IV. *G. Newell* (Providence, R.I.).

Salmon, J. Etude des plasmas en régime transitoire. J. Phys. Radium (8) **17** (1956), 931–933.

This paper deals with almost the same problem as that of the preceding review. The present paper makes a further specialization to a gas in which the collision rate is independent of velocity. It is shown that if the distribution of electrons is initially Maxwellian, it will remain Maxwellian with a time dependent "temperature" if the surrounding molecules have a Maxwellian velocity distribution at another fixed temperature. The temperature difference decays exponentially with time. It is also shown that if the density of electrons is non-uniform, the density satisfies the classical diffusion equation with a time dependent diffusion parameter. *G. Newell.*

Hosemann, R.; Bonart, R.; und Schoknecht, G. Faltungspolynom und Gitterfaktor parakristalliner Gitterwerke. Z. Physik **146** (1956), 588–614.

Eine Erweiterung der Theorie des idealen Parakristalles auf beliebig geformte Gitterzellen wird gegeben. Sie entartet in die 1950 gegebene Form, wenn diese Gitterzellen Parallelepipede sind. Andernfalls tritt eine Korrelationskorrektur auf, die im Bereich der starken Reflexe und im Bereich der Gasinterferenzen vernachlässigbar ist. Im Bereich der diffusen Interferenzmaxima dagegen ver-

mittelt sie Einblicke in die Korrelationsverhältnisse innerhalb einer Elementarzelle. Im Gegensatz zur Kristalltheorie gibt es unter allen möglichen einfach-primitiven Gitterzellen eine einzige, die sich dadurch vor allen anderen auszeichnet, dass in ihr die Korrelationsverhältnisse am stärksten sind. Dieses ist die parakristalline Elementarzalle. Die Häufigkeitsstatistiken ihrer Kanten- und Diagonalvektoren hängt von den Nahordnungsverhältnissen im Parakristall ab. — Die Elementarzelle ist aus dem Faltungsquadrat der Struktur dadurch eindeutig erkennbar, dass ihre Kantenstatistiken eine minimale Schwankungsbreite haben. Sie ist aber auch aus den Eigenschaften der Intensitätsfunktion eindeutig errechenbar. — Die Gesamtstatistik $z^1(x)$ ist eindeutig als Faltungspolynom von 4 bzw. 13 Grundstatistiken im zweidimensionalen bzw. im dreidimensionalen Gitter entwickelbar. Dadurch erst gelingt eine Berechnung der Intensitätsfunktion. Es wird zum Ausdruck gebracht, dass die vorliegende geometrische Theorie der Parakristalle die mathematisch einfachste ist. Sie enthält die geometrische Theorie der Kristalle von v. Laue und diejenige der Flüssigkeiten von Zernicke-Prins und Debye als entartete Sonderfälle. — Das Faltungspolynom wird an einer zweidimensionalen parakristallinen Punktstruktur durch lichtoptische und numerische Entfaltungen nachgeprüft. Innerhalb der statistischen Fehler besteht gute Uebereinstimmung zwischen Theorie und Experiment. *W. Nowacki* (Bern).

Bardeen, J. Interaction between electrons and lattice vibrations. Canad. J. Phys. **34** (1956), 1171–1189.

This paper presented at the International Conference on Electron Transport in Metals and Solids (Ottawa, September 1956) gives a survey and comparison of various approximations used to describe the interaction potential between electrons and lattice vibrations particularly in metals. It is mainly a review of the work that has been done and the present status of problems rather than a comprehensive self-contained description of various theories. The scope is limited to models and their physical justification with relatively little discussion of the use and consequence of the models in theories of electrical resistance, superconductivity, etc. *G. Newell.*

★ **Lifshits, I. M.; and Kosevich, A. M. Theory of the De Haas-van Alphen effect for particles with arbitrary dispersion law.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1957. 6 pp.

Translated from Dokl. Akad. Nauk SSSR (N.S.). **96** (1954), 963–966.

Park, David. A summation method for crystal statistics. Physica 22 (1956), 932–940.

Because the magnetization of the two-dimensional Ising lattice is known to involve irrational algebraic functions, the author conjectures that the same may be true of some properties of other lattices for which several terms of various power series expansions are known. Attempts are made to fit such power series expansions to functions of the type $s(x)=\prod_i (1+C_i x)^{k_i}$ in which there are only a few factors. Although the author does not succeed in reproducing exactly the coefficients in any of the expansions tested except the one that was known to be algebraic, he does obtain surprisingly accurate approximations to them with only a few factors. The techniques used for approximating the series are simple enough so as to be potentially useful in extrapolating other functions which one suspects are at least approximately of the above form. *G. Newell* (Providence, R.I.).

Frank, D. Zur Statistik der Spinwellen. Z. Physik **146** (1956), 615–628.

Für einen Kristall wird im Sinne der Heitler-London-Näherung die Austauschwechselwirkung behandelt. Entgegen der bisherigen Auffassung wird gezeigt, dass die Spinwellen ein Fermi-Gas mit Wechselwirkung sind, wenn die Atome ausserhalb abgeschlossener Schalen nur je ein Elektron besitzen. Um dies darzulegen, wird einmal die Möglichkeit benutzt, aus dem Symmetriecharakter der Wellenfunktion auf das statistische Verhalten zu schliessen. Die exakte Eigenfunktion der linearen Kette wird aufgeschrieben und begründet. Sie ist antisymmetrisch in den Wellenzahlen. Andererseits führt die Behandlung der Austauschwechselwirkung mit der zweiten Quantelung zu dem gleichen Ergebnis. Von entscheidender Bedeutung ist in beiden Beweisen der Ausschluss polarer Zustände. Ausserdem erhält man zwangsläufig Aussdrücke, die in diesem Fall an die Stelle der Formeln von Holstein und Primakoff treten. Mit den üblichen Vernachlässigungen lässt sich auch für Flächengitter und lineare Kette eine spontane Magnetisierung errechnen. *W. Nowacki.*

Segall, B. Calculation of the band structure of "complex" crystals. Phys. Rev. (2) **105** (1957), 108–115.

A method for studying the band structure of "complex" crystals (i.e., crystals having more than one atom per unit cell) is developed. This method is a generalization of one proposed independently and arrived at by different approaches by Korringa and Kohn and Rostoker for the study of the band structure of "simple" crystals. The general approach leads to a promising method when the crystalline potential can reasonably be approximated by a potential which is spherically symmetric within non-overlapping spheres about the lattice sites and is constant elsewhere. Important virtues of the method are its expected accuracy and the fact that the largest part of the labor involved is in the computation of certain "structure constants" which are applicable to all crystals with the same crystallographic structure. *W. Nowacki.*

Bauer, Ernest. Coupling of optic and acoustic modes of vibration in crystals. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **CX-28** (1956), i+22 pp.

The transition probability for transfer of energy from one optic to two or three acoustic modes of vibration has been calculated for NaCl, and found to be of the order 10^{12} per sec at low temperatures. *W. Nowacki.*

Attree, R. W.; and Plaskett, J. S. The self-energy and interaction energy of stacking faults in metals. Phil. Mag. (8) 1 (1956), 885–911.

A model is considered of a crystal lattice having slip of less than a lattice vector on two close packed planes. The effect of this slip on the eigenvalues of one-electron wave functions extending throughout the crystal is calculated, using first order perturbation theory to give the wave functions in a perfect crystal. The self-energy of a fault in a monovalent metal is found to be about 20 ergs/cm². The interaction energy is found to be about 0.4 ergs/cm² when the faults are one lattice plane apart and negligible when they are further apart. An unsuccessful attempt is made to calculate the effect of electrons with wave vectors near zone faces. The theory

involved a clarification of the Bloch-Floquet theory [Bloch, Z. Physik **52** (1928), 555–600; Floquet, Ann. Sci. Ecole Norm. Sup. (2) **12** (1883), 47–88] and the use of a contour integral to sum the zeros of an exponential sum.
W. Nowacki (Bern).

Hearmon, R. F. S. The elastic constants of anisotropic materials. II. Advances in Physics 5 (1956), 323–382 (1 plate).
In Rev. Mod. Phys. **18** (1946), 409–436 the author summarized work which had been published to the end of 1944 on the elastic constants of anisotropic materials, mainly crystals. This summary is now brought up to date (1955) by a review of work published since that time. The scope of the review is indicated by the headings of each section. 1. Theory: General Theory, Notations and Nomenclature. Third Order Elastic Coefficients. Atomic and Lattice Theory of the Elastic Constants. 2. Experimental: Methods of Measurement. Results. Effect of Temperature. Effect of Stress. Effect of Radiations. Effect of Electrical Conditions. 3. Applications: Polycrystalline Aggregates. Wave Propagation and the Debye Temperature. Miscellaneous. *A. E. Green.*

See also: Mackenzie, p. 602; Phillips, Rogers and Wilson, p. 605; Wilson, p. 605; Cowan and Ashkin, p. 625.

Elasticity, Visco-elasticity, Plasticity

Mustafaev, A. A. Axis-symmetric loading of elastic half-space. Akad. Nauk Azerbaĭdžan. SSR. Dokl. **12** (1956), 319–324. (Russian. Azerbaijani summary)
The problem of an elastic semi-space loaded by a concentrated force applied at a point inside the space at a finite distance from the boundary was solved in a closed form by R. D. Mindlin [Physics, **7** (1936), 195–202]. Expressions for stress components derived by Mindlin are complicated and not very convenient for numerical computations.
The author of this paper gives another solution of this problem which is more suitable for practical applications. He finds a stress function which satisfies the biharmonic equation and the boundary conditions and derives from it the stress components. Then he shows that the Boussinesque solution for a force applied at the boundary and the Kelvin solution for a force applied at an infinitely distant point inside the space are special cases of his solution. *T. Leser* (Aberdeen, Md.).

Duffin, R. J. Analytic continuation in elasticity. J. Rational Mech. Anal. **5** (1956), 939–950.
Fundamental for elasticity theory and hydrodynamics of viscous fluid flow in space is the system of partial differential equations for four functions u_1, u_2, u_3 and f consisting of the three equations $\Delta u_i = \partial f/\partial x_i$ $(i=1, 2, 3)$ and $\sum_i \partial u_i/\partial x_i = -cf$, c being a constant. The author shows first that the solution must be analytic and he derives, for $c \neq -1$, a new representation of the solutions in terms of harmonic functions. Using the representation he obtains several theorems regarding analytic continuation across plane boundary parts of the domain where the functions are defined. We quote one of these theorems explicitly: Let E be a domain symmetric with respect to $x_1=0$ and (u_i, f) a solution of the above equation in the part of E where $x_1>0$. Assume further that the u_i have boundary values θ on $x_1=0$. Then the solution can be analytically continued into the whole E by the formulas

$$u_1' = -u_1 + x_1\left(2cf + 2\frac{\partial u_1}{\partial x_1} - x\frac{\partial f}{\partial x_1}\right)(1+2c)^{-1},$$

$$u_2' = -u_2 + x_1\left(-2\frac{\partial u_1}{\partial x_2} + x_1\frac{\partial f}{\partial x_2}\right)(1+2c)^{-1},$$

$$u_3' = -u_3 + x_1\left(-2\frac{\partial u}{\partial x_3} + x_1\frac{\partial f}{\partial x_3}\right)(1+2c)^{-1},$$

$$f' = \left(f - 2cf - 4\frac{\partial u}{\partial x_1} + 2x_1\frac{\partial f}{\partial x_1}\right)(1+2c)^{-1}.$$

Here $u'_i(x_1, x_2, x_3) = u_i(-x_1, x_2, x_3)$ and $f'(x_1, x_2, x_3) = f(-x_1, x_2, x_3)$, and it is assumed that $c \neq -1$ or $-\frac{1}{2}$. Similar theorems are obtained where the boundary condition on $x_1=0$ contains derivatives. *C. Loewner.*

Teodorescu, Petre P. About a general method of solving the plane problem of elastodynamics. Com. Acad. R. P. Romîne **6** (1956), 795–801. (Romanian. Russian and English summaries)
L'auteur obtient une seule fonction de tension pour la détermination des composants de tension et le déplacement vectoriel pour quelques problèmes plans de la théorie d'élasticité. Le cas dynamique est réduit à la solution d'un équation du type

$$[\Delta + k_{n1}][\Delta + k_{n2}]F_n(x, y) = 0.$$

Cependant, l'auteur ne présente pas une solution d'un problème particulier avec les conditions aux limits, donc, l'analyse doit être considerée seulement théorétique.
K. Bhagwandin (Oslo).

Schaefer, H. Die drei Spannungsfunktionen des zweidimensionalen ebenen Kontinuums. Österreich. Ing.-Arch. **10** (1956), 267–277.
Considering a two-dimensional continuum as a limiting case of a three-dimensional continuum, the author obtains three stress functions, one of which is Airy's stress function for plane stress. Although the author is under the impression that the other two have not yet been used, they are in fact the stress functions introduced in 1941 by Fox and Southwell in a restricted war-time report, which was republished in Philos. Trans. Roy. Soc. London. Ser. A. **239** (1945), 419–460 [MR **7**, 268]. In addition to discussing the use of these stress functions in the theory of elastic plates along lines similar to those suggested by Fox and Southwell, the author indicates a novel application to plane rigid frames loaded perpendicularly to their plane. This yields an elegant derivation and promises an appreciable extension of the analogy of Prager and Hay [Quart. Appl. Math. **1** (1943), 49–60; MR **4**, 233].
W. Prager (Providence, R.I.).

Netrebko, V. P. Torsion of an elastic parallelepiped. Vestnik Moskov. Univ. **11** (1956), no. 6, 11–25. (Russian)
This paper is closely related to another by the same author [same Vestnik **9** (1954), no. 12, 15–26; MR **16**, 881].
The boundary conditions are: displacements zero on one base of the rectangular parallelepiped, given distribution of shearing stress or given displacements on the other. A variational method is again used. Two numerical examples are treated in considerable detail. The stress distributions differ little from Saint Venant's except near

the ends (distant less than the greatest linear dimension of the end). *R. C. T. Smith* (Armidale).

Sarkisyan, M. S. Bending of a prismatic rod of double-t cross section. Akad. Nauk Armyan. SSR. Izv. Fiz.-Mat. Estest. Tehn. Nauki **9** (1956), no. 7, 61–77. (Russian. Armenian summary)

From symmetry about the x- and y-axes it is only necessary to consider a quarter of the cross-section, e.g. the region consisting of two overlapping rectangles

(I) $0 \leq x \leq b_1,\ 0 \leq y \leq d_1,$

(II) $b_1 - d_2 \leq x \leq b_1,\ 0 \leq y \leq b_2,$

where $b_1 > d_2$, $b_2 > d_1$.

The stress function (which by the Saint Venant theory satisfies Poisson's equation and simple boundary conditions) is given by different expressions F_1, F_2 in the regions I, II, these series being found by separation of variables. Making F_1 agree with F_2 for $x = b_1 - d_2$, $0 \leq y \leq d_1$ (part of the boundary of II but a line inside I) and for $b_1 - d_2 \leq x \leq b_1$, $y = d_1$, leads to a completely regular system of equations in infinitely many unknowns.

The results show (a) that the tangential stresses are greatest on the horizontal axis of symmetry, (b) Žuravski's formula (for X_z) is reasonably exact only for points on the wall of the sections, (c) the stresses are almost independent of the value of Poisson's ratio. *R. C. T. Smith.*

Bishop, R. E. D. The analysis of vibrating systems which embody beams in flexure. Proc. Inst. Mech. Engrs. **169** (1955), 1031–1046, discussion 1046–1050.

The paper is concerned with the use of receptance functions in analysing the flexural vibrations of uniform beams. It contains, in essence, a development of a method whose potentialities were first pointed out by W. J. Duncan [Phil. Mag. (7) **32** (1941), 401–409; 34 (1943), 49–63; MR **4**, 179] and B. C. Carter [Aero. Res. Comm., Rep. and Memo. no. **1988** (1941)]. The purpose of the paper is 1) to explain in detail the use of receptances in beam flexure, 2) to indicate by means of examples the very large range of problems which can be tackled with receptances, and 3) to present tables of the functions involved. These tables provide a ready means of handling many complicated practical systems. In particular, they may be used in the analysis of free and forced vibration of frames, but this type of system is not introduced in the paper, as frame vibration problems represent a large subject which can more conveniently be dealt with separately.

Receptance functions have been discussed by several authors [Duncan, ibid. no. **2000** (1947); MR **9**, 109; D. C. Johnson, Engineering **171** (1951), 650–652; R. E. D. Bishop, J. Roy. Aero. Soc. **58** (1954), 703–719]. The nature of the receptance functions is described at some length in the last-mentioned paper where several of their properties are listed in the form of tables. (The term "receptance" is used here in accordance with published suggestion that a previous name should be discarded owing to the possibility of confusion with electrical terminology). *R. Gran Olsson* (Trondheim).

Kosmodamianskiĭ, A. S. Bending of an anisotropic beam under the action of a uniform load. Rostov. Gos. Univ. Uč. Zap. Fiz.-Mat. Fak. **32** (1955), no. 4, 75–94. (Russian)

The author analyses a three-dimensional problem of beams of constant cross-section under constant distributed load. The beams are homogeneous, anisotropic, and through every point inside a beam passes a plane of elastic symmetry normal to its geometric axis which coincides with the z axis. Deflections are assumed to be small. The author assumes the bending moment to be a quadratic function of z with three undetermined coefficients which are found later from the boundary conditions. The stress components are derived from two stress functions satisfying boundary conditions and the equilibrium and compatibility equations. After the general analysis the author investigates the following cases of boundary conditions: 1) cantilever beam, 2) one end fixed another simply supported, 3) both ends fixed. In all three cases he derives expressions for the deflection curve, for the maximum deflection, and for the curvature, and compares them with the corresponding formulas from the elementary theory. In cases 1) and 2) the first term in each formula represents the simple formula used in the strength of materials theory, in case 3) the author's formulas and the elementary theory formulas are identical. The author investigates further a beam of elliptic cross-section and also an orthotropic beam of rectangular cross-section. The paper ends with conclusions where the author states the conditions when the elementary theory gives satisfactory approximations and when it does not. *T. Leser* (Aberdeen, Md.).

Mustafaev, A. A. Bending of single flexible foundation elements. Akad. Nauk Azerbaĭdžan. SSR. Dokl. **12** (1956), 163–168. (Russian. Azerbaijani summary)

The solution of the differential equation for an elastically supported beam

$$[EI\frac{d^4y}{dx^4} + \varkappa(x)y = 0$$

with bending moment and shear applied at $x=0$, is expressed as an infinite series of repeated integrals. For $\varkappa(x)$ discontinuous of the form $\varkappa_n(x) = a_n + \alpha_n x$, $h_n < x < h_{n+1}$ (a_n, α_n constant), a power series for $y(x)$ is obtained whose coefficients involve $a_1, \cdots, a_j, \alpha_1, \cdots, \alpha_j$ for $h_j < x < h_{j+1}$. *R. C. T. Smith* (Armidale).

Gorgidze, A. Ya. On the problem of mutual influence of the bending of a beam by a transverse force on the bending by a couple. Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **22** (1956), 251–263. (Russian)

Finite strain theory is used to discuss the problem stated in the title. The beam is composite but Poisson's ratio is assumed to be the same in all portions of the beam. Each displacement is written as the sum of the displacement according to classical linear theory and an additional term with the product of the couple and the transverse force as coefficient. By neglecting second and higher order terms in the additional displacements, nonhomogeneous linear equations are obtained for the stress function and stresses corresponding to these additional displacements. *R. C. T. Smith* (Armidale).

van der Linden, C. A. M. Thermal stresses in a plate containing two circular holes of equal radius, the boundaries of which are kept at different temperatures. Appl. Sci. Res. A. **6** (1956), 117–128.

The problem dealt with in this paper originated in an examination of thermal stresses in cylinder heads, containing two openings through which gases, with considerable differences in temperature, pass. As the cylinder

head is a multiply-connected system, viz. a plate with two holes, the temperature can induce thermal stresses, even in the stationary case. As is usual in calculating stresses in the neighbourhood of holes, the region in which they occur can be considered infinite. Since the purpose of the paper is the investigation mainly of the stress condition between the holes, the influence of the outer boundary can be neglected.

In the calculation bipolar coordinates were introduced as treated for stress calculation by G. B. Jeffery [Philos. Trans. Roy. Soc. London. Ser. A. **221** (1920), 265–293]. The method for taking into account the temperature in the calculation is that usually applied [E. Melan and H. Parkus, Wärmespannungen infolge stationärer Temperaturfelder, Springer, Wien, 1953, p. 25; MR **16**, 306]. For the notation of strains and stresses the tensor method is used.

As pointed out by the author it is difficult to judge how far the results can be applied to the stresses in a cylinder head. The stiffness and the temperature of the cylinder wall and of the pipes in the holes are playing a part. Furthermore the cylinder head itself is not of uniform thickness. For this reason the calculated stresses will not be more than an indication of the magnitude of the real stresses. The results of the calculation are therefore no proof, but only an acceptable explanation that the thermal stresses are able to cause difficulties. The stresses are so large as to be considered a grave danger.

R. Gran Olsson (Trondheim).

Guest, J. The compressive buckling of a clamped parallelogram plate with a longitudinal stiffener along the centre-line. Austral. J. Appl. Sci. **7** (1956), 191–198.

The paper follows an investigation into an unstiffened parallelogram plate and its purpose is to determine whether any important changes occur when the clamped plate is supported by a central stiffener. The problem as indicated in the title has been studied in some detail for one particular case. Because the differential equation of the deflection w for the clamped edge condition cannot be exactly solved, i.e. in a closed form, the method of Galerkin was employed to obtain the buckling load in the direction of the applied stress. For the deflection of the plate the functions of S. Iguchi [Eine Lösung für die Berechnung der biegsamen rechteckigen Platten, Springer, Berlin, 1933] with a slight modification were found practicable. The modified functions are

$$F_p(\xi)=\xi(\xi-1)^2+(-1)^p\xi^2(\xi-1)-\sin(p\pi\xi)/p\pi,$$
$$G_q(\eta)=\eta(\eta-1)^2+(-1)^q\eta^2(\eta-1)-\sin(q\pi\eta)/q\pi,$$

((ξ,η) being non-dissensional oblique coordinates) which satisfy the boundary conditions $w=0$, $\partial w/\partial\xi=0$ for $\xi=0$, $\xi=1$; $w=0$, $\partial w/\partial\eta=0$ for $\eta=0$, $\eta=1$. The author has found that for the particular elastic parameters $\gamma=B/bD$, $\delta^*=P/bt\sigma_x$ chosen, the least buckling load is associated with a symmetric buckling mode. (B=flexural rigidity of the stiffener, D=flexural rigidity of the plate, P= compressive force in stiffener, σ_x=critical compressive stress in the force direction, b and t=width resp. thickness of the plate.)

The results obtained are presented graphically and compared with those of W. H. Wittrick [Aero. Quart. **4** (1953), 151–163; MR **14**, 927] for the unstiffened plate under the same constraints.

R. Gran Olsson.

Levi, Franco. Sul calcolo degli effetti di bordo nelle vòlte sottili cilindriche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 342–346.

The different sets of approximations used in various thin shell theories eventually lead to similar values for the externally observable quantities. The author essays a qualitative explanation of this fact. *D. R. Bland.*

Ambartsumyan, S. A. On the theory of anisotropic shallow shells. NACA Tech. Memo. no. **1424** (1956), 11 pp.

A translation of the Russian article which appeared in Akad. Nauk SSSR. Prikl. Mat. Meh. **12** (1948), 75–80 and was reviewed in MR **10**, 87.

Ambartsumyan, S. A. On the calculation of shallow shells. NACA Tech. Memo. no. **1425** (1956), 11 pp.

A translation of the Russian article which appeared in Akad. Nauk SSSR. Prikl. Mat. Meh. **11** (1947), 527–532 and was reviewed in MR **9**, 397.

Nazarov, A. A. On the theory of thin shallow shells. NACA Tech. Memo. no. **1426** (1956), 7 pp.

A translation of the Russian article reviewed in MR **11**, 486.

★ **Kopzon, G. I. Vibration of thin-walled elastic bodies in a gas flow.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 6 pp.

Translated from Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 217–220. The original Russian article was reviewed in MR **18**, 163.

★ **Kopzon, G. I. Vibrations of a shallow wing-shell in a gas flow.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 6 pp.

Translated from Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 377–380. The original Russian article was reviewed in MR **18**, 526.

Bunič, L. M.; Paliĭ, O. M.; and Piskovitina, I. A. Stability of a truncated conical shell under uniform external pressure. Inžen. Sb. **23** (1956), 89–93. (Russian)

The stability problem of a conical shell solved by the energy method was presented by É. I. Grigolyuk [Inzen. Sb. **19** (1954), 73–82]. The authors of this paper use a similar method to solve the stability problem of a truncated conical shell. The displacements which satisfy boundary conditions after the loss of stability are assumed and the total energy of the system is derived by using these assumed displacements. The expressions for displacements contain three undetermined parameters which are found by taking partial derivatives of the energy expression with respect to each parameter and setting them equal to zero. The formula for the critical pressure which the authors derive is valid for cylindrical and conical shells as well. Let r_0 and r_1 be the radii at the bases and 2α be the angle at the vertex. Setting $r_0=r_1$ and $\alpha=0$ the authors' formula becomes the von Mises formula for cylindrical shells; on setting $r_0=0$ tpe authors' formula does not become Grigolyuk's formula for conical shells (reference as above) but deviations from it turn out to be very small. *T. Leser* (Aberdeen, Md.).

Itow, Tomio. Elastic and plastic state of stress around a deep circular shaft. Tech. Rep. Osaka Univ. **5** (1955), 349–355.

Consider a circular shaft of radius R and center 0

driven vertically (z-direction) into the ground. The author assumes circular symmetry, so that the shaft is surrounded by an annular plastic region and this in turn by the elastic region. The problem is to determine the depth to which the shaft (without a lining) can be sunk without collapsing. The following additional assumptions are made: (1) the stresses in the elastic region are, $\sigma_r=kr^{-2}+sz$, $\sigma_\theta=-kr^{-2}+sz$, $\sigma_z=wz$, where k, s, w are constants; (2) the stress, σ_z, in the plastic region is $\sigma_z=wz$; (3) the Coulomb yield condition is valid in the plastic region; (4) the r, θ, z, are the principal directions of stress in both the elastic and plastic regions. These stresses satisfy the equilibrium relations and the elastic compatibility relations (due to the linearity of the stresses in z). By substituting $\sigma_z=wz$ and $\sigma_r=0$ (along the shaft) into the yield condition, a quadratic relation in σ_θ is obtained. The vanishing of the discriminant furnishes the desired condition on z. Further, the elastic-plastic boundary radius is determined. The reviewer feels that assumptions (2) and (4) are too strong. *N. Coburn.*

Marin, Joseph. Mechanical properties of materials for combined stresses based upon true stress and strain. J. Franklin Inst. 263 (1957), 34–46.

The well-known difference between the load-extension and the stress-strain diagram (called the "true" diagram) in uni-axial tension is discussed; an assumed power-function for the latter is used to derive various "time" material characteristics. The proposed extension to three-dimensional conditions disregards the difficulties arising from the necessity of considering finite strains under conditions other than "radial" loading, as well as of redefining the concept of "ultimate stress".
A. M. Freudenthal (New York, N.Y.).

Thomas, T. Y. Slip surfaces in plastic solids. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 923–927.

The author discusses conditions holding at a surface at which the velocity vector and stress tensor have a finite discontinuity, the normal component of velocity and stress vector being continuous. Assuming the stress is plane and satisfies a quadratic yield condition, but employing no stress-strain relations, he obtains an equation relating the direction of the discontinuity surface to the stresses on both sides of the surface, then discusses simple tension. His footnote 5 is misleading, it being always possible to take $\theta=0$ or $\frac{1}{4}\pi$ in equation (13) by suitably choosing the coordinates. He seems to have in mind a special coordinate system, such as in used in equation (14). *J. L. Ericksen* (Washington, D.C.).

See also: Hersch, p. 579; Tanimura, p. 581; Ginzburg, p. 619.

Fluid Mechanics, Acoustics

★**Crank, J. The mathematics of diffusion.** Oxford, at the Clarendon Press, 1956. vii+347 pp. $8.00.

In diesem Buch macht der Verfasser einen ernsten Versuch eine einheitliche mathematische Darstellung von dem schnellwachsenden Feld der Diffusionstheorie zu geben. Die grösste Aufmerksamkeit verdienen die Beiträge die im Laufe der letzten Jahre gemacht worden sind: allerdings gibt der Verfasser in den ersten sechs Kapiteln nur allgemeine Lösungen von fast allen wesentlichen Problemen mit dem passendem Rand- und Anfangsbedingungen (z.B. die Diffusion durch ebene Platten, Zylinder, Kugeln und halb-ebene Bereiche). Die rEgebnisse sind durch zahlreiche Figuren erläutert. Die linearen Ergebnisse sind von grosser Wichtigkeit zur Lösung von einigen sehr komplizierten nicht linearen partiellen Differentialgleichungen zweiter Stufe, die in der Diffusionstheorie auftreten. Diese Lösungen darf man oft als Anfangswerte in den numerischen Lösungsprozessen (besonders in den Iterationsprozessen) benutzen. Im siebenten Kapitel behandelt der Verfasser die Diffusionsprozesse mit einer beweglichen Grenze (so wie z.B. wenn eine Flüssigkeit die eine Komponente von einer Gasmischung absorbiert, wenn eine Flüssigkeit fortschreitend in den festen Zustand übergeht, oder wenn der Diffusionskoeffizient nicht kontinuierlich ist). Die mathematischen Lösungsmethoden werden P. V. Danckwerts zugeschrieben [Trans. Faraday Soc. **46** (1950), 701–712; MR **12**, 264]. Die genauen Lösungen einiger Aufgaben dieser Kategorie werden auch gegeben.

Im achten Kapitel werden die Prozesse der gleichzeitigen Diffusion und chemische Reaktion behandelt. Verdunstung von einer Oberfläche verschiedener Körper und die Fälle umkehrbarer Reaktion werden ebenfalls diskutiert [s. J. Crank, Phil. Mag. (7) **39** (1948), 362–376; MR **9**, 591]. Im neunten Kapitel behandelt der Verfasser Probleme mit variablen Diffusionskoeffizienten. Zugleich werden konzentrationsabhängige Diffusion in gewissen unbegrenzten Medium studiert. Neben der Boltzmannschen Transformation werden auch Iterationsverfahren angewandt: beispielsweise werden Diffusionskoeffizienten exponentieller und linearer Art einer genaueren Untersuchung unterworfen. Der übrigbleibende Teil des Kapitels ist den sehr anwendungsfähigen Verfahren von H. Fujita [Text. Res. J. **22** (1952), 757–760, 823–827] gewidmet. Bekanntlich ist diese Methode die einzige welche analytische Lösungen von nicht linearen partiellen Differentialgleichungen dieser Art liefern kann. Einige Lösungen werden graphisch illustriert. Die von H. Yamada entwickelte Methode von 'Momenten' [Rep. Res. Inst. Fluid Engrg., Kyûsyû Univ. **3** (1947), no. 3, 29–35] welche H. Fujita [Mem. Coll. Agric., Kyoto Univ. no. **59** (1951), 31] zur Lösung nicht linearer Gleichungen anwendet, wird auch dargestellt. Diese Methode ist sehr wirkungsvoll zur Lösung verschiedener komplizierter Probleme. Im zehnten Kapitel werden die numerische Lösungsmethoden diskutiert. Fast alle anwendbare Methoden werden untersucht. Im elften Kapitel werden die verschiedenen Definitionen und Messungsmethoden von Diffusionskoeffizienten behandelt. Sowohl theoretische als experimentelle Fragen kommen in Betracht. Im zwölften Kapitel gibt der Verfasser numerische und graphische Resultate für Probleme mit variablen Diffusionskoeffizienten. Im dreizehnten und letzten Kapitel werden Prozesse mit gleichzeitiger Diffusion von Wärme und Feuchtigkeit behandelt. Die explizite Lösung ist im Falle einer ebene Platte in Dampf dargestellt.

Sprachlich ist das Buch sehr gut. Interessierte Fachleute werden es sicherlich jahrelang benutzen. Zurzeit gibt es in keiner anderen Sprache ein ebenbürtiges Buch.

Leider hat der Verfasser über Grundwasserströmung und Diffusion von Neutronen nichts gesagt, obwohl in diesen Gebieten schon eine umfangsreiche Literatur vorhanden ist. *K. Bhagwandin* (Oslo).

Gheorghiev, Gh. Quelques aspects géométriques du mouvement permanent d'un fluide idéal. Rev. Univ. "Al. I. Cuza" Inst. Politehn. Iaşi 2 (1955), 43–64. (Romanian. Russian and French summaries)

When $\phi=\frac{1}{2}v^2-U+\int\rho^{-1}dp$ and the fluid is "baro-

tropic" ($\rho=f(p)$), then $\bar{v}\times\text{rot}\,\bar{v}=\text{grad}\,\phi$. The surfaces S_B with equation $\phi=$const have been called surfaces of Bernoulli by B. Caldonazzi (Boll. Un. Mat. Ital. 4, 1–3 (1925)), see also S. S. Byušgens, Izv. Akad. Nauk SSSR. Ser. Mat. **12** (1948), 481–512; MR **10**, 633]. In the present paper we find characteristics for the S_B and especially for the cases 1) div $\bar{v}=0$, 2) rot $\bar{v}\times\bar{v}=0$, 3) $\bar{v}\cdot\text{rot}\,\bar{v}=0$, 4) streamlines are asymptotic lines. Conditions are derived that a family of surfaces can be the support for an infinity of ideal currents; for this it is necessary that they be parallel, that the net of streamlines and their orthogonal trajectories be isothermic, and that the surface gradient of the differences of the normal curvatures and geodesic torsions of the curves of the net be collinear with its Čebyšev vector. Some remarks are made on two notes by V. Vîlcovici on barotropic fluids [Acad. R.P. Române. Bul. Şti. Şect. Şti. Mat. Fiz. **4** (1952), 541–545; **5** (1953), 147–154; MR **15**, 754; **16**, 1168]. *D. J. Struik.*

Graffi, Dario. Il teorema di unicità per i fluidi incompressibili, perfetti, eterogenei. Rev. Un. Mat. Argentina **17** (1955), 73–77 (1956).

L'auteur établit un théorème d'unicité relatif au problème aux limites posé par le mouvement d'un fluide parfait, incompressible et hétérogène dans un domaine fini D limité par une surface σ. Supposons données: a) à tout instant $t>0$ la force massive comme fonction de la densité ρ, du point P (point courant dans D) et de t; on suppose que cette fonction a une dérivée finie par rapport à ρ; b) à l'instant initial la densité ρ et la vitesse **V** dans tout le domaine D; c) à tout instant $t>0$ la composante normale de **V** sur la surface σ et d'autre part les autres composantes de **V** et la densité ρ en tous les points de σ où **V** est dirigé vers l'intérieur de D. L'auteur démontre qu'il ne peut exister plus d'un système de valeurs de la vitesse **V**, de la densité ρ et de la pression p (cette dernière déterminée à une fonction arbitraire de t près) satisfaisant aux conditions ci-dessus et déterminées en tout point P de D et à tout instant $t>0$. Pour démontrer ce théorème l'auteur utilise entre autres certaines inégalités qu'il avait établies dans un travail précédent [J. Rational Mech. Anal. **2** (1953), 99–106; MR **14**, 598]. Le théorème peut être étendu à un domaine infini à condition de faire certaines hypothèses sur l'allure à l'infini de ρ, p et **V**. *R. Berker* (Istanbul).

De, S. C. Kinematic wave theory of bottlenecks of varying capacity. Proc. Cambridge Philos. Soc. **52** (1956), 564–572.

This is a development of the general kinematic wave theory of Whitham and the reviewer [Proc. Roy. Soc. London. Ser. A. **229** (1955), 281–316, 317–345; MR **17**, 309, 310], with applications to flood movement and traffic flow. We assumed a relationship between flow (quantity passing per unit time), concentration (quantity per unit distance) and position in a one-dimensional system. The present author supposes, more generally, a relationship varying with time. In particular he studies traffic flow through a bottleneck, when the capacity of the bottleneck gradually falls to a value below that required by the constant oncoming flow and later rises to a value above it again. Expressions are found for the "hold-up time" during which a slow crawl is present ahead of the bottleneck. *M. J. Lighthill* (Manchester).

Dorrestein, R. Theory of "kinematic" waves. Nederl. Tijdschr. Natuurk. **22** (1956), 270–276. (Dutch)

The paper is mainly expository. The author comments upon wave problems connected with the equation

$$\frac{\partial q}{\partial t}+c(x,q)\frac{\partial q}{\partial x}=0,$$

describing, e.g., high-water waves down a river or traffic waves on a one-way road [M. J. Lighthill and G. B. Whitham, Proc. Roy. Soc. London. Ser. A. **229** (1955), 281–316, 317–345; MR **17**, 309, 310]. The author suggests the term "frictional waves" instead of Lighthill and Whitham's "kinematical waves". *C. J. Bouwkamp.*

Hansen, Arthur G.; and Herzig, Howard Z. On possible similarity solutions for three-dimensional incompressible laminar boundary layers. I. Similarity with respect to stationary rectangular coordinates. NACA Tech. Note no. **3768** (1956), 30 pp.

Für dreidimensionale, inkompressible, laminare Grenzschichten, werden alle ähnlichen Lösungen bei stationärem, rechtwinkligem Koordinatensystem x, y, z gesucht. Zu diesem Zweck werden mit gruppentheoretischen Betrachtungen alle Parameter $\eta=\eta(x,y,z)$ bestimmt, die die partiellen Grenzschicht Diff. Gl. in gewöhnliche Diff. Gl. überführen. Die möglichen Parameter η sowie die zugehörigen Strömungen ausserhalb der Grenzschicht werden in einer Tabelle zusammengestellt und die Eigenschaften von Aussenströmung und Grenzschicht diskutiert. Die sich bei den ähnlichen Lösungen ergebenden gewöhnlichen Diff. Gl. werden nicht numerisch ausgewertet, es wird gegebenenfalls nur auf bereits bekannte Lösungen hingewiesen. *L. Speidel* (Molheim).

Hawthorne, W. R.; and Armstrong, W. D. Shear flow through a cascade. Aero. Quart. **7** (1956), 247–274.

Es wird die Strömung hinter einem ebenen Schaufelgitter bei ungleichförmiger Zuströmung untersucht, wobei die Ungleichförmigkeit auf eine Geschwindigkeitsänderung in Spannweitenrichtung der Schaufeln beschränkt bleibt. Für die Behandlung dieser Aufgabe wird das Gitter durch eine Wirkungsebene ersetzt, ein Verfahren, das ähnlich auch in der Propellertheorie angewandt wird. Für die Strömung durch eine derartige Wirkungsebene werden dann die allgemeinen Bewegungsgleichungen für reibungslose, inkompressible Flüssigkeit aufgestellt. Diese Bewegungsgleichungen werden für den vorliegenden Fall noch durch Vernachlässigung kleiner Glieder vereinfacht, so dass die Strömung hinter dem Schaufelgitter bei ungleichförmiger Zuströmung ermittelt werden kann. Bei Beispielrechnungen wird der Umfang der Störung, der mittlere Abströmwinkel sowie die Strömungsumlenkung verändert. Ein Vergleich mit Messungen zeigt gute Übereinstimmung. *L. Speidel* (Molheim).

Fabri, Jean; et Siestrunck, Raymond. Sur le calcul des petites perturbations propagées à son apparition par le décollement tournant d'une roue axiale. C. R. Acad. Sci. Paris **243** (1956), 1718–1721.

The problem of "rotating stall" in axial flow machines is attacked by a linear small-perturbation scheme. The entire flow pattern consisting of a periodic succession of stalled and unstalled parts is assumed to be rotating steadily. Different relations between upstream and downstream flow variables are written down for the unstalled and the stalled portions of the circumference, respectively. This leads to a boundary-value problem involving these two different boundary conditions on the circumference of the unit circle. In the solution it is then required that the local angle of attack of the blades have a fixed, critical,

value at both ends of each stalled region. This leads to an expression for the propagation speed which depends on the exit angle of the cascade (assumed independent of entrance conditions) and the critical angle of attack, but not on any other properties of the blade-characteristic relation. *W. R. Sears* (Ithaca, N.Y.).

Carafoli, E. Sur la théorie des profils à contour donné. Acad. R. P. Romåne. Bul. Şti. A. 1 (1949), 513–520. (Romanian. Russian and French summaries)

Let P be a profile defined by $y=y(x)$ in the $z=x+iy$ plane, let $A'B'$ be its chord of length c, and let K_1 be a circle with center at the midpoint of $A'B'$ and of radius $\frac{1}{4}c$. To approximate steady incompressible flow about P the author approximates the conformal mapping of P onto K_1 by

$$(*)\qquad z=(1+\mu)\zeta_1e^{-i\tau}+\sum_0^n e^{-2i\tau}q_j[e^{-i\tau}/(1+\mu)\zeta_1]^j.$$

Consider also the mapping of K_1 onto $A'B'$. By assuming that two points on P and $A'B'$ with the same abscissas correspond approximately to the same point $\zeta_1=ae^{i\theta}$ of K_1 the author determines q_j to make (*) fit P at n points. The q_j can be expressed as simple functions of Fourier coefficients of $y(x)$. μ and the angle of zero lift, τ, are approximated by simple integrals, and lift and moment coefficients of P are calculated. For Joukowski airfoils the formulas for μ and τ are exact. *J. Giese.*

Wolska, J. Sur une solution de l'équation du mouvement permanent du fluide visqueux. Ann. Polon. Math. 3 (1956), 13–18.

By means of a suitable change of variables, the author transforms the equation for the stream function of a stadey plane motion of an incompressible viscous fluid,

$$\frac{\partial\Psi}{\partial y}\frac{\partial\Delta\Psi}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial\Delta\Psi}{\partial y}=\nu\Delta\Delta\Psi$$

into a system of four non-linear integral equations in four unknowns. Using the method of successive approximations, a family of local solutions is constructed, depending on four arbitrary analytic functions of a complex variable. *R. Finn* (Pasadena, Calif.).

Lane, C. A. Viscous dissipation caused by a sphere. J. Acoust. Soc. Amer. 28 (1956), 1194–1196.

The author estimates contributions to the viscous energy dissipation associated with early terms of a perturbation expansion for the stream function of an oscillating sphere. The oscillation amplitude is assumed small compared with the sphere radius, a, and also $a(\omega/2\nu)^{\frac{1}{2}}\gg 1$, where ω denotes the frequency and ν the kinematic viscosity. It is concluded that the classical dissipation (primary a.c. vortices) due to the leading term, outweighs the other contributions (including d.c. vortices). Boundary dissipation effects are omitted.

H. Levine (Palo Alto, Calif.).

Sanyal, Lakshmi. On the flow of a viscous liquid between two co-axial circular cylinders under a periodic pressure-gradient. J. Tech., Calcutta 1 (1956), 43–47.

The one-dimensional flow of a viscous liquid between two coaxial circular cylinders under a periodic pressure gradient is considered. It is shown that when the period is very small the motion at each instant is like the steady laminar motion corresponding to the instantaneous pressure gradient, while when the period is very large the motion has the boundary-layer character. (From author's summary.) *D. W. Dunn* (Ottawa, Ont.).

Carrier, G. F.; and Di Prima, R. C. On the torsional oscillations of a solid sphere in a viscous fluid. J. Appl. Mech. 23 (1956), 601–605.

Most treatments of the torsional oscillations of solid bodies assume that the velocity field is circumferential. In this paper the motion in planes containing the axis of oscillation is also considered. An expansion in terms of the angular displacement ε (assumed small) is made. The first approximation to the circumferential velocity is computed, and then used in computing the first approximation to the pumping motion. This is used to compute the correction to the circumferential velocity and, in particular, the correction to the viscous torque. For the range of parameters considered it is found that the correction to the torque is of the order of $0.04\varepsilon^2|N_0|$, where N_0 is the classical viscous torque. This problem is of interest in practical viscosity measurements. (From authors' summary.) *A. E. Green* (Newcastle-on-Tyne).

Saffman, P. G. On the motion of small spheroidal particles in a viscous liquid. J. Fluid Mech. 1 (1956), 540–553.

Small spheroidal particles suspended in a sheared viscous liquid sometimes take up slowly preferred orientations, relative to the motion of the undisturbed liquid, which are independent of the initial conditions. These observations cannot be accounted for by the solution of the linearized Navier-Stokes equations. The author shows that the effect of the inertia of the liquid alters slowly the orbit of the particle in accordance with Jeffery's hypothesis that the particle ultimately moves in such a way that the dissipation of energy is a minimum, but that this effect is orders of magnitude too small to account for any of the observations.

The author suggests that non-Newtonian properties of the liquid account for the observations. He shows that the rate of orientation of a particle is independent of its size, if the non-Newtonian terms are small, a prediction which is verified experimentally. Other experimental evidence in support of this suggestion is described. *A. E. Green.*

Proudman, Ian. The almost-rigid rotation of viscous fluid between concentric spheres. J. Fluid Mech. 1 (1956), 505–516.

Viscous fluid is contained between two concentric spheres which rotate about the same axis with angular velocities Ω and $\Omega(1+\varepsilon)$, where ε is very small compared with unity. The flow is regarded as a small perturbation superimposed upon a rigid body rotation of the fluid as a whole. The equations are linearized in the magnitude of the perturbation, the validity of the linearization being independent of the Reynolds number of the primary rotation. Attention is then restricted to the case in which the Reynolds number is large.

Author shows that the cylindrical surface that touches the inner sphere (the axis being the axis of rotation) is a singular surface in which velocity gradients are large. Outside this cylinder the fluid rotates as a rigid body. Inside the cylinder the velocity distribution in the central core of the motion is determined by the velocity distribution in the boundary layers over the spheres. The mechanics of the cylindrical shear layer itself is discussed.

A. E. Green (Newcastle-on-Tyne).

Shercliff, J. A. The flow of conducting fluids in circular pipes under transverse magnetic fields. J. Fluid Mech. 1 (1956), 644–666.

For the final fully developed laminar flow, the solution is obtained when the walls are conducting but without contact resistance. When the fluid conductivity and field strength are sufficiently high, boundary layers occur with a thickness inversely proportional to normal field intensity. The induced potential difference (across the pipe perpendicular to the field) is then 0.926 of the value corresponding to uniform velocity, if the walls are non-conducting. For laminar flow in the entry region in the case of uniform inlet velocity and non-conducting walls, the solution is obtained by means of a Rayleigh approximation. The entry length is shorter than for a rectangular pipe, but is still too long for appreciable distortion of the velocity profile in practical flowmeters except at low flow rates. The corresponding pressure drop is independent of field strength, if this is high, and is about one-eigth of the drop in the non-conducting case. Experiments are described which confirm the theoretical results for fully developed flow within the limits of experimental accuracy, and which show that the results for the entry region are correct in order of magnitude. *D. W. Dunn.*

Dobryšman, E. M. Approximate solution of some non-stationary problems of the boundary layer. Prikl. Mat. Meh. **20** (1956), 402–410. (Russian)

In the boundary layer of a two-dimensional incompressible flow about a profile P let the curves x=const be lines normal to P, and let y=const be their orthogonal trajectories, with $y=0$ on P. Let u (or v) be the x (or y)-component of velocity, and suppose that $u=f(t,x)$ at the edge of the boundary layer $y=\delta(t,x)$. Then

$$\partial f/\partial t+f\partial f/\partial x=\rho^{-1}\partial p/\partial x.$$

Eliminate v and p between this and the boundary layer equations to obtain the integro-partial differential equation

$$(*)\qquad \frac{\partial^2 u}{\partial y^2}=\frac{\partial u}{\partial t}+\frac{u\partial u}{\partial x}-\frac{\partial u}{\partial y}\int_0^y\frac{\partial u}{\partial x}\,dy-\frac{\partial f}{\partial t}-f\frac{\partial f}{\partial x}.$$

The author writes $u=u^{(0)}+u^{(1)}+\cdots$, chooses

$$u^{(0)}(t,x,y)=yf(t,x)/\delta(t,x),$$

and substitutes $u^{(0)}$ ($u^{(1)}$) in the right (left) member of (*) to define $u^{(1)}$. Then $u=u^{(0)}+u^{(1)}$ is forced to satisfy $u=0$ at $y=0$ and $u=f$ at $y=\delta$. To assure smooth transition from viscous to non-viscous flow he demands $\partial u/\partial y=0$ at $y=\delta$, which implies that $\varphi=f\delta^2$ satisfies

$$(**)\qquad \frac{8}{3f}\frac{\partial\varphi}{\partial t}+\frac{\partial\varphi}{\partial x}+5\varphi\partial\log\frac{f}{\partial x}=16.$$

Explicit solutions φ are found for f of the following types: $f=f_1(t)f_2(x)$; $f=x+f_1(t)$; $f=t^n$. They include examples of decay (growth) of Blasius flow over plates in decelerated (impulsively started) motion. An analog of (**) is also found for axisymmetric flow. *J. H. Giese.*

Craya, Antoine. Sur les corrélations triples en trois points en turbulence homogène. C. R. Acad. Sci. Paris **244** (1957), 560–562.

L'auteur montre que, si $u_i(\mathbf{x})$ est la vitesse d'agitation d'un fluide incompressible, en turbulence homogène, le tenseur

$$\phi_{ijk}(\mathbf{k},\mathbf{k}')=\frac{i}{(2\pi)^6}\int\overline{u_i(\mathbf{x})u_j(\mathbf{x}+\mathbf{r})u_k(\mathbf{x}+\mathbf{r}')}d\mathbf{r}d\mathbf{r}',$$

transformé de Fourier du tenseur des corrélations triples en trois points, s'exprime à l'aide de quatre scalaires indépendants (qui se réduisent à deux si la turbulence est isotrope). *J. Bass* (Paris).

Moore, Franklin K.; and Ostrach, Simon. Average properties of compressible laminar boundary layer on flat plate with unsteady flight velocity. NACA Tech. Note no. **3886** (1956), 1+35 pp.

In this paper the authors apply the methods which they have previously established [Moore, NACA Tech. Note **2471** (1951); MR **13**, 401; Ostrach, ibid. no. **3569** (1955)] to the problem of calculating the time-average properties of a nearly quasi-steady boundary layer. This requires new second-order solutions of their differential equations, which they find by a method identical with that of their previous works. They obtain interesting results for the average skin friction, heat transfer rate and boundary layer displacement thickness. *K. N. C. Bray.*

Ginzburg, I. P. Hydraulic shock in pipes of elasto-viscous material. Vestnik Leningrad. Univ. **11** (1956), no. 13, 99–108. (Russian)

The author derives a partial differential equation and specifies initial and boundary conditions for the mass flow of a compressible fluid with linear pressure-density relation in one-dimensional unsteady motion in a pipe. For elasto-viscous (relaxing) pipes the equation is of fourth (third) order. For pipes of constant cross section and constant thickness the order is decreased by one. To show the damping effect of the viscosity of the material of which the pipe is composed on wave propagation the author uses Laplace transforms to construct an approximate solution for visco-elastic pipes of constant cross section. *J. H. Giese* (Aberdeen, Md.).

Karpman, Gilbert. Onde de compression dans un fluide contenu dans un tore rigide à section carrée. C. R. Acad. Sci. Paris **243** (1956), 770–773.

L'auteur étudie l'onde de compression stationnaire se propageant dans un fluide parfait qui remplit un tore à section carrée. Il considère les solutions fondamentales de l'équation de l'onde; le rayon de gorge du tore étant supposé très petit, la partie des solutions fondamentales qui présente une singularité à l'origine (fonction de Bessel de seconde espèce) est omise. L'auteur calcule l'énergie potentielle et l'énergie cinétique de l'onde pour les premières solutions fondamentales ($n=2$ à 22). Chacun des niveaux d'énergie obtenus n'est pas nécessairement stable par rapport à des perturbations extérieures. Dans un prochain travail l'auteur compte étudier cet aspect du problème. *R. Berker* (Istanbul).

Wanner, Marcel. Propriétés des écoulements unidimensionels permanents d'un gaz quelconque dans une tuyère de section variable ou non avec échange de chaleur et dissipation d'énergie due à la viscosité. C. R. Acad. Sci. Paris **243** (1956), 1485–1487.

The author derives some general results concerning the one-dimensional flow of an arbitrary gas through a nozzle of variable section with heat exchange and viscous dissipation. *D. W. Dunn* (Ottawa, Ont.).

Schulz, Gerhard. Die Wirksamkeit von Wölbungsklappen in Überschallbereich. Z. Flugwiss. **5** (1957), 15–22.

Die durch einen Klappenausschlag am Flügel entstehenden zusätzlichen Kräfte und Momente werden nach

der linearisierten Überschalltheorie berechnet. Die Berechnung erstreckt sich auf den Auftrieb und die Klappenlast, das Kippmoment und das Betätigungsmoment, das Rollmoment, den induzierten Widerstand sowie das induzierte Giermoment. Die in allgemeiner Form gewonnenen Ergebnisse werden auf den Deltaflügel mit Klappe angewandt. *Zusammenfassung des Autors.*

Zel'dovič, Ya. B. Motion of a gas under the action of a momentary pressure (shock). Akust. Ž. 2 (1956), 28–38. (Russian)

At time $t=0$ a shock tube is half filled with gas at rest so that the density has constant value $\rho=\rho_0$ when $x>0$ and $\rho=0$ when $x<0$, where x is the distance along the tube. Initially the values of temperature and pressure are zero. A pressure pulse starts from the interface and moves into the dense part of the tube. The strength of the pulse $\pi(t)$ is zero when $t=0$, reaches a maximum value P and then decays. A time of decay τ is defined such that $\pi(t)\ll P$ when $t\gg\tau$.

The resulting gas motion is determined with use of similarity hypotheses. In particular it is shown that if X defines the position of the pulse at time t its strength π decays according to the law $\pi\sim X^{-n}$ where $1<n<2$ and $n=1.33$ for a polytropic gas with $\gamma=1.4$. *M. Holt.*

Lunev, V. V. Laminar boundary layer of compressible gas on a plate in the case of large temperature changes. Prikl. Mat. Meh. 20 (1956), 395–401. (Russian)

The equations of compressible boundary layer flow are expressed as a system of two non-linear partial differential equations for the specific enthalpy i and the shear stress $\tau=\mu\partial u/\partial y$ as functions of the coordinate x and u, the x-component of velocity, parallel to the plate. If $\rho\mu/\rho_\infty\mu_\infty=f(i)$, then by separation of variables particular solutions $i=i(u)$, $\tau=X(x)g(u)$ can be found. In fact (1) $gg''=-f(i)u$, and (2) $(i'/\sigma)'-(g'/g)(1-1/\sigma)i'+c=0$, where c is a constant, and the Prandtl number σ is constant or slowly varying. The author writes

$$g=(f_{cp})^{\frac{1}{2}}(g_*+g_1+g_2+\cdots),$$

where f_{cp} is a mean value of f, and g_* is a solution of (1) for constant f. Then $g_1, g_2, \cdots$ can be found successively, if at the same time successive approximations to solutions i of (2) are calculated. By perturbing (1) and by taking advantage of the fact that $f(i)$ is slowly varying the author finds that the error $(\delta g)/g_0=\frac{1}{2}m(f(i_0), u)$, where m is a mean value of the error $(\delta f(i))/f_0$, and g_0 and i_0 are exact solutions of (1) and (2). From this he shows that the errors in the nth approximations to g_0 and i_0 decrease exponentially. It is said that for most computations it will suffice to determine g_1 or at most g_2. *J. H. Giese.*

Lomax, Harvard; and Heaslet, Max. A. Recent developments in the theory of wing-body wave drag. J. Aero. Sci. 23 (1956), 1061–1074.

The theoretical study of wave drag on airplanes flying at supersonic speeds is re-examined in the light of the recent discovery of the transonic area rule. A supersonic area rule is presented together with a discussion of its range of applicability. In regions where the supersonic area rule is seriously in error other methods, based on linearised theory, for calculating the wing body interference effects are outlined. In addition, some recent results based on the nonlinear small perturbation equation for transonic flow are briefly discussed. Comparison between theory and experiment are given. (Authors' summary.) *R. M. Morris* (Cardiff).

Kondo, Kazuo. On the approximate expression of the theory of the lifting line. J. Fac. Engrg. Univ. Tokyo 24 (1955), no. 3, 1–19.

Für die Lösung der Integralgleichung der tragenden Linie (Bestimmung der Zirkulationsverteilung eines Tragflügels bei gegebenen Abmessungen) wird ein Näherungsverfahren entwickelt, das eine Verallgemeinerung der von H. Multhopp [Luftfahrtforschung 15 (1938), 153–169] angegebenen Näherungslösung darstellt. Um die in der Nähe von Unstetigkeitsstellen auftretende geringere Genauigkeit zu verbessern, wird eine Verteilungsfunktion eingeführt, die die Aufpunkte zur Bestimmung der Zirkulationsverteilung in der Umgebung der Unstetigkeit besonders dicht legt. Die Lösung entspricht dann der des Multhopp'schen Verfahrens, nur dass die Koeffizienten, die dort universell berechnet werden konnten, jetzt von der gewählten Verteilungsfunktion abhängen. Für die Berechnung der Koeffizienten werden einfache Beziehungen angegeben. Das Verfahren wird für einige elliptische Flügel mit strengen Lösungen verglichen und zeigt gute Übereinstimmung. *L. Speidel* (Mülheim/Ruhr).

★**Carman, P. C. Flow of gases through porous media.** Academic Press Inc., New York; Butterworths Scientific Publications, London, 1956. ix+182 pp. $6.00.

Le but de l'auteur est de donner un exposé des problèmes relatives à l'écoulement des gazes à travers des milieux poreux, attention particulière étant faite aux contributions pendant les derniers quinze ans.

Cependant, l'auteur ne présente pas un exposé unifié de ces phénomènes. Un tel exposé demanderait nécessairement une analyse mathématique plus soigneuse, basée sur l'intuition physique. L'exposé de l'auteur est virtuellement depourvu de principles de recherches physico-mathématiques. Par exemple, l'auteur seulement fait mention du mot "boundary-condition".

De l'autre côté les aspects experimentaux et chimiques sont traités a fond. De temps en temps des publications théoriques sont mentionées.

Malheureusement, l'auteur surveille plusieures contributions rélatives aux écoulements en milieux poreuse, par exemple, les travaux de Litwiniszyn, Polubarinova-Kočina, etc. (voir Mathematical Reviews, Zentralblatt für Mathematik, Referativnyĭ Žurnal (Mehanika), etc.).

Cependant, du point de vue chimique et technique ce livre est une contribution très valable. Dans le premier chapitre l'auteur présente l'équation de Kozeny-Carman. Les limitations de ces types d'équations sont mentionées. Le deuxième chapitre est consacré à l'écoulement visqueux. Quelques problèmes rélatives a texture anormale sont étudiés. Dans le troisième chapitre l'auteur étudie le phénomène de glissement, l'écoulement libre de molécules etc. L'auteur fait mention aussi de quelques contributions importantes de B. Derjaguin [C. R. (Dokl.) Acad. Sci. URSS (N.S.) 53 (1946), 623–626]. Les chapitres quatre, cinq et six sont consacrés à la mesure de l'aire de surface au moyen des méthodes de permeabilité, l'écoulement des gazes sorbable, et la séparation des mixtures de gaz, respectivement. Le septième chapitre (le dernier) est consacré a quelques particulier modes de l'écoulement de gaz. *K. Bhagwandin* (Oslo).

Chambré, Paul L. On chemical surface reactions in hydrodynamic flows. Appl. Sci. Res. A. 6 (1956), 97–113.

Es wird die reibungsbehaftete, stationäre Strömung von heterogenen Flüssigkeiten untersucht, die an der

Begrenzungswand katalytisch reagieren, so dass die Begrenzungswand selbst unverändert bleibt. Es wird vorausgesetzt, dass die reagierenden Anteile der Flüssigkeit klein sind, die Flüssigkeitseigenschaften von der Reaktion oder der Erwärmung also nicht beinflusst werden. Die Wandtemperatur wird konstant gehalten. Die exakte mathematische Formulierung des Problems führt auf ein System von Integralgleichungen. Entsprechende Integralgleichungen lassen sich beim konvektiven Wärmeübergang in erzwungener Strömung bei veränderlicher Wandtemperatur formulieren, so dass die für diese Strömungen bekannten Lösungen übernommen werden können. *L. Speidel* (Mülheim/Ruhr).

See also: Kopzon, p. 615; Romanov, p. 624; Sestapalov, p. 625; Simon, p. 628.

Optics, Electromagnetic Theory, Circuits

★**Maréchal, André. Optique géométrique générale.** Handbuch der Physik. Bd. 47, pp. 44–170. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956.

An extremely well organized and well written exposition of geometric optics. The proofs are, in general, elegant and informative. The notational difficulties almost endemic to the subject are avoided.

Here, in outline, are the subjects treated. A. General laws of geometric optics: I. Generalities; II. Geometric optics deduced from Maxwell's equations; III. Fermat's principle; IV. Geometric optics and electronic optics. B. Rigorous investigation of stigmatism: I. General method; II. Stigmatic refractive surfaces and mirrors; III. Correction plates, optical retouching; IV. Media having a continuous variation of index, application of certain optical systems to short waves; V. Investigation of stigmatism in an element of volume. C. Gaussian approximation: I. Definitions, notations and sign conventions; II. Constituent elements, the refractive surface and the mirror; III. Image formation in centered systems; IV. Optical homography, the impossibility of perfect optical systems. D. The field: I. The lateral field; II. The depth of field and depth of focus. E. The aberrations: I. Generalities; II. Chromatic aberrations; III. Geometric aberrations. F. Image contrast, influence of aberrations: I. The formation of images of extended objects; II. Influence of aberrations on contrast. G. Conclusion. There are some references and less than a page of bibliography.

Especially noteworthy is the report of recently developed techniques of Fourier analysis for evaluating the quality of images formed by optical instruments. Radar workers will be interested in the discussion of the Luneberg lens. *G. L. Walker* (Southbridge, Mass.).

Altrichter, O.; und Schäfer, Gerta. Herleitung der Gullstrandschen Grundgleichungen für schiefe Strahlenbüschel aus den Hauptkrümmungen der Wellenfläche. Optik 13 (1956), 241–253.

This result was previously derived in a more direct way by Juan L. Rayces [J. Opt. Soc. Amer. 43 (1953), 705–706]. *G. L. Walker* (Southbridge, Mass.).

Focke, Joachim. Wellenoptische Untersuchungen zum Öffnungsfehler. Opt. Acta 3 (1956), 110–126.

In order to apply the wave theory of light to practical optical systems it is shown that solutions of the wave equation can be found in the image space so that the surfaces of equal phase agree asymptotically with the wave surfaces of geometrical optics at distances far from the Gaussian focus.

The theory is applied to calculate the intensity variation along the optical axis and also in various assumed image planes in the case of a point source imaged by an optical system having no aberrations other than spherical aberration. The results appear to agree reasonably well with experimentally measured values. *E. W. Marchand.*

Horiuchi, Kazuo. Electromagnetic fields due to current flowing parallel to interface of two different media. J. Phys. Soc. Japan 12 (1957), 170–176.

The author considers the two-dimensional problem of determining the scattered field of an electromagnetic wave incident upon a conducting half-plane which lies along the flat interface of two different media. He shows that if the incident wave is either plane and polarized parallel to the interface or cylindrical and produced by a line source parallel to the edge of the half-plane, the current induced in the half-plane satisfies an integral equation of the Wiener-Hopf type. *C. H. Papas.*

★**Klimontovich, Iu. L. To the new phenomenological theory of superconductivity.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 5 pp.

Translated from Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 384–386.

★**Klimontovich, Iu. L. On the diamagnetism of superconductors.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 5 pp.

Translation from Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 44–46.

★**Turbovich, I. T. Certain generalization of the Kotel'nikov theorem.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 12 pp.

Translated from Radiotehnika **11**, no. 4, 1956, pp. 5–14.

King, Jean I. F. Radiative equilibrium of a line-absorbing atmosphere. II. Astrophys. J. **124** (1956), 272–297.

The radiative equilibrium of a plane-parallel, nongray atmosphere of finite thickness, (τ_1) heated solely from below by uniform isotropic radiation, is investigated with a view to determining the thermal structure and the angular distribution of the emergent radiation. The problem is equivalent to solving the integrodifferential equation

$$(*)\quad \mu\lambda\frac{dI(\tau,\mu\lambda)}{d\tau}=I(\tau,\mu\lambda)-\frac{1}{2}\int_{-1}^{1}\int_{0}^{1}I(\tau,\mu\lambda)\frac{d(\nu/\Delta\nu)}{\lambda}\,d\mu,$$

where $1/\lambda=\kappa_\nu/\bar{\kappa}$ is the ratio of the monochromatic to the mean absorption coefficient and the rest of the symbols have their standard meanings. By extending the method of the principles of invariance [Chandrasekhar, Radiative transfer, Oxford, 1950, chap. X; MR 13, 136], the author shows that the required solution can be expressed in terms of X- and Y-functions which are now defined by the equations

$$X(\mu\lambda)=$$
$$1+\frac{\mu\lambda}{2}\int_{0}^{1}\int_{0}^{1}\frac{X(\mu\lambda)X(\mu'\lambda')-Y(\mu\lambda)Y(\mu'\lambda')}{\mu\lambda+\mu'\lambda'}\,d\mu'\,\frac{d(\nu'/\Delta\nu)}{\lambda},$$

$$Y(\mu\lambda)=e^{-\tau_1/\mu\lambda}+\frac{\mu\lambda}{2}\int_0^1\int_0^1\frac{Y(\mu\lambda)X(\mu'\lambda')-X(\mu\lambda)Y(\mu'\lambda')}{\mu\lambda-\mu'\lambda'}d\mu'\frac{d(\nu'/\Delta\nu)}{\lambda}.$$

Thus

$$I(0,\mu\lambda)=\tfrac{1}{2}I_g\beta_0[X(\mu\lambda)+Y(\mu\lambda)],$$

$$I(\tau_1,-\mu\lambda)=I_g\{1-\tfrac{1}{2}\beta_0[X(\mu\lambda)+Y(\mu\lambda)]\},$$

where $I_g=F[\beta_0(\alpha_1+\beta_1)]^{-1}$, F the constant net flux and α_n and β_n moments defined by

$$\alpha_n=\int_0^1\int_0^1 X(\mu\lambda)(\mu\lambda)^n d\mu\frac{d(\nu/\Delta\nu)}{\lambda},$$

$$\beta_n=\int_0^1\int_0^1 Y(\mu\lambda)\,(\mu\lambda)^n d\mu\,\frac{d(\nu/\Delta\nu)}{\lambda}.$$

The equation of radiative transfer (*) is also solved explicitly by the method of discrete ordinates and the elimination of the constants is carried out completely in a manner analogous to that described by Chandrasekhar [Astrophys. J. **106** (1947), 152–216; MR **9**, 444]. For the case [cf. King, ibid. **121** (1955), 711–719; MR **16**, 1161]

$$\lambda=\frac{\cosh\beta-\cos(2\pi\nu/d)}{\sinh\beta},$$

where β is a constant, explicit numerical solutions for the appropriate H- and X- and Y-functions are tabulated for various values of the parameters which are of practical interest. The paper represents an important advance in radiative transfer theory and it is not possible to do justice to it in a short review. *S. Chandrasekhar.*

Whitney, Charles. Stellar pulsation. I. Momentum transfer by compression waves of finite amplitude. Ann. Astrophys. **19** (1956), 34–43. (French and Russian summaries)

The propagation of one-dimensional waves of finite amplitude in an isothermal plane-parallel atmosphere in which the density follows the barometric law is considered. The equations of motion and continuity are reduced to the standard Riemann form

$$(*)\qquad \frac{\partial}{\partial\tau}(v\pm\log\rho+\tau)+(v\pm1)\frac{\partial}{\partial x}(v\pm\log\rho+\tau)=0,$$

where x, τ and v are the coordinate, time and velocity measured in suitable units. At a shock front the condition $\rho_2/\rho_1=\sigma^2$ must be satisfied where σ is the velocity of shock-propagation in the same units as v. An initial state of rest is assumed; and at $t=0$ the particle at $x=0$ starts describing a periodic motion with a prescribed amplitude and frequency. Equation (*) is then integrated numerically. The solutions derived are exhibited in a series of graphs. Using the results of his integrations the author defines an "effective gravity" in terms of the difference $\bar{p}-\frac{1}{2}\overline{\rho v^2}$, where the mean pressure $\bar{p}$ and the mean square kinetic energy $\frac{1}{2}\overline{\rho v^2}$ are over the basic period.

S. Chandrasekhar (Williams Bay, Wis.).

Manarini, Anna Marisa. Sulla velocità dell'energia elettromagnetica nei cristalli in moto. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 357–362.

For plane waves in a moving crystal the velocity of electromagnetic energy is defined as

$$\mathbf{C}_e=\frac{2c\mathbf{E}\times\mathbf{H}}{\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B}}.$$

Let $\mathbf{C}_e'$ be the corresponding velocity in the rest frame of the crystal. The paper shows that the transformation connecting $\mathbf{C}_e$ and $\mathbf{C}_e'$ is the same as that connecting the velocities $\mathbf{v}$ and $\mathbf{v}'$ of a moving particle. For simplicity, the calculations are made only for the case of a crystal moving along one of its dielectric axes. *J. L. Synge.*

★ **Iakovlev, O. I. Taking antenna height into account in the theory of tropospheric scattering of meter-waves.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1957. 7 pp.

Translated from Radiotekh. i Elektr. **1** (1956) No. 3, 309–312.

Toraldo di Francia, Giuliano. Momento di quantità di moto ceduto da un'onda elettromagnetica a un piccolo ellissoide, avente conduttività unidirezionale. Boll. Un. Mat. Ital. (3) **11** (1956), 332–343.

This paper is concerned with the determination of the cross-section of absorption of moment of momentum from elliptically polarised plane electromagnetic waves by an ellipsoid which has unidirectional conductivity and whose linear dimensions are small compared with the wave-length of the incident radiation.

As preliminaries, certain questions relating to unidirectional conductors are discussed. The relation between the cross-section for scattering and the above cross-section for absorption is considered; as also is the electrostatics of unidirectional conductors.

E. T. Copson (St. Andrews).

Mushiake, Yasuto. Backscattering for abitrary angles of incidence of a plane electromagnetic wave on a perfectly conducting spheroid with small eccentricity. J. Appl. Phys. **27** (1956), 1549–1556.

"The scattering of a plane electromagnetic wave by a perfectly conducting spheroid with small eccentricity is treated by a method of expanding the scattered field directly in a series of spherical vector wave functions. An expression for the first-order solution of the back-scattered field is obtained for arbitrary angles of incidence of a plane wave. The numerical values of echo areas computed from the first-order expression are shown for various cases, and rough experimental results are also discussed. During this study some mathematical formulas involving the associated Legendre functions have been derived." (Author's summary.) *S. Karp.*

Morgan, S. P.; and Young, J. A. Helix waveguide. Bell System Tech. J. **35** (1956), 1347–1384.

The author considers a special waveguide structure consisting of a helix of conducting wire with a lossy dielectric in the interspace. Using the sheath helix approximation, he shows that for a given circular waveguide mode, there is a helix pitch which allows a circularly polarized version of the mode to propagate with little attenuation. The wires, in this case, follow the current lines. Particular interest is shown in the TM_{01} mode which requires a helix of essentially zero pitch and in the other TM_{0n} modes which are attenuated strongly by such a structure. Extensive computations are presented for various values of loss in the interspace dielectric.

It appears to the reviewer that such a helix structure wound for the TE_{11} might also be used to advantage as an outer conductor for a ferrite isolator of the Faraday rotation type. *W. K. Saunders* (Washington, D.C.).

Chambers, Ll. G. Propagation in a ferrite-filled waveguide. Quart. J. Mech. Appl. Math. 8 (1955), 435–447.

A solution is obtained for acomp letely filled rectangular waveguide by perturbation techniques. No calculations are presented. It appears that perturbation may be applicable since the guide is completely filled and the unperturbed field is taken as that of a guide, filled with a material of high dieletric constant. [For a case, however, in which the perturbation technique is not applicable, see B. Lax, K. J. Button and L. M. Roth, J. Appl. Phys. 25 (1954), 1413–1421.] *W. K. Saunders.*

★**Berk, A. D. Cavities and wave-guides with inhomogeneous and anisotropic media.** Massachusetts Institute of Technology, Research Laboratory of Electronics, 1955. ii+58 pp.

This paper which was originally submitted as a thesis in June 1954 treats some problems of wave-guides and of cavities filled with anisotropic media. Both exact and perturbation techniques are employed. The author takes the modal solutions suggested by Slater [Microwave electronics, Van Nostrand, New York, 1950] and extends them to include anisotropic media, and to avoid a completeness counterexample suggested by Teichmann and Wigner [J. Appl. Phys. 24 (1953), 262–267; MR 14, 823]. He now states, but does not prove, that the set is complete. He compares his method of expansion to that of Schwinger [Radiation Lab. Rep., Mass. Inst. Tech. 43–34 (1943)] and discusses the points of difference in Appendix 1.

His problems include: the transmission cavity with ferrite sphere, input impedance of the reaction type cavity with ferrite sphere, rectangular wave-guide with dielectric slab, circular wave-guide with coaxial dielectric core, circular wave-guide with thin dielectric sliver, circular wave-guide with coaxial ferrite core, ferrite slab in rectangular wave-guide, and several others.

Some general variational formulas for the propagation numbers in wave-guide and for the resonant frequencies of cavities are developed.

A bibliography inadvertently omitted in the binding of the report is being forwarded to the recipients under separate cover.

{Note: The reviewer feels that the restriction to hermitian quantities in (97) is probably unnecessary. It is common practice to use formulas similar to (97) to find losses by observing changes in the Q of the cavity at resonance. This requires that μ be regarded as other than hermitian.} *W. K. Saunders* (Washington, D.C.).

Stanyukovič, K. P. Some results in the field of relativistic magneto-gas dynamics. Dokl. Akad. Nauk SSSR (N.S.) 103 (1955), 73–76. (Russian)

The equations of motion of relativistic magneto-gas dynamics can be written in Minkowski 4-space in the form

$$(1)\qquad \frac{\partial}{\partial x_k}(T_{ik}+\overline{T}_{ik})=0,$$

where T_{ik} denotes the electromagnetic stress tensors and $\overline{T}_{ik}$ the material stress tensor:

$$(2)\qquad \overline{T}_{ik}=(p+\varepsilon)u_iu_k+p\delta_{ik},$$

where u_i is the four-velocity, $\varepsilon=\rho c^2$ is the energy, the density including internal energy and p the pressure. We also have Maxwell's equations. For the one-dimensional case

$$u_1=u_1(x_1)=u,\ u_2=u_3=0,$$
$$H_3=H_3(x_1)=H,\ H_2=H_1=0,$$

the equations are specialized and it is shown that for adiabatic flow there exists the integral

$$H/\rho=\text{const.}$$

On using this integral the equations of motion and continuity can be reduced to the canonical form

$$(3)\qquad \frac{\partial}{\partial t}(q\pm\Omega)+\frac{u+\omega^*}{1\pm u\omega^*/c^2}\frac{\partial}{\partial x}(q\pm\Omega)=0,$$

where

$$dq=\frac{du}{1-u^2/c^2},\ d\Omega=+\omega^*d\log\rho,$$

$$\omega^*=c\left\{\frac{d(p+H^2/8\pi)}{d(\varepsilon+H^2/8\pi)}\right\}^{1/2}.$$

Equation (3) is similar to Riemann's equation for waves of finite amplitude in ordinary gas dynamics. The author obtains various special solutions of this equation. The Hugoniot equations governing the propagation of direct shocks also are derived [cf. de Hoffmann and Teller, Phys. Rev. (2) 80 (1950), 692–703; MR 12, 769].

S. Chandrasekhar (Williams Bay, Wis.).

Stanyukovič, K. P. Elements of relativistic magneto-gas-dynamics. Izv. Akad. Nauk SSSR. Ser. Fiz. 19 (1955), 639–650. (Russian)

A more complete discussion of the results briefly derived in the paper reviewed above.

S. Chandrasekhar (Williams Bay, Wis.).

Kaplan, S. A.; and Stanyukovič, K. P. On the solution of inhomogeneous problems of one-dimensional motion in magneto-gas-dynamics. Ž. Eksper. Teoret. Fiz. 30 (1956), 382–385; supplement to 30, no. 2, 8. (Russian. English summary)

In an earlier paper [Dokl. Akad. Nauk SSSR (N.S.) 95 (1954), 769–771; MR 15, 1001] the authors have shown that one dimensional waves of finite amplitude in hydromagnetics are governed by the equations

$$(1)\qquad \frac{\partial u}{\partial t}+\frac{\partial}{\partial h}\left(p_g+\frac{\mu H^2}{8\pi}\right)=0,\ \frac{\partial u}{\partial h}=\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right),$$

where $h=\int_0^x\rho dx$, $H=H(x)$ is the magnetic field, p_g is the gas pressure and the rest of the symbols have their usual meanings. Also

$$(2)\qquad H/\rho=b(h),$$

where b is a function of the argument specified. On the assumption

$$p=p_g+\frac{\mu H^2}{8\pi}=\left[\frac{A^2k}{\rho^{-1}-\sigma(h)}\right]^k\left(\frac{h_0}{h+h_0}\right)^{3k-1},$$

where A, k and h_0 are constants and $\sigma(h)$ is some function of h, one can obtain special solutions of (1). Thus with the substitutions

$$\tau=(h+h_0)^{-1},\ z=p\tau,\ w=u/\tau+\int pdt$$

the equations to be solved reduce to

$$Ah_0^{(3k-1)/k}z^{-(1+k)/2k}\partial X_\pm/\partial t+\partial X_\pm/\partial\tau=0,$$

where

$$X_{\pm}=w\pm\frac{2kA}{k-1}\,h_0^{(3k-1)/2k}z^{(k-1)/2k}.$$

A solution of this equation is

$$X_{\pm}=\text{constant}. \tag{4}$$

Various special solutions of equation (1) which follow from (4) above are discussed. *S. Chandrasekhar.*

Nardini, Renato. Su un tipo di onde magneto-idrodinamiche non omogenee. Boll. Un. Mat. Ital. (3) **11** (1956), 350–358.

This is a continuation of previous work by the same author [same Boll. (3) **10** (1955), 349–362; MR **17**, 325]. He now considers the case when the fluid is non-viscous and perfectly conducting, so that, in the notation of the review quoted, ν and $1/\gamma$ are zero. The equations are solved under the following simplifying assumptions. (a) The magnetic field **H** is assumed to have components

$$H_r=0,\ H_\theta,\ H_z=H_0(r)$$

in cylindrical coordinates, where $H_0(r)$ is a given function of r alone. It follows that H_θ is independent of θ. (b) The velocity **v** of the fluid has components

$$v_r=0,\ v_\theta,\ v_z=0,$$

where v_θ must be independent of θ. (c) p and U are independent of θ.

The equations then simplify, and it follows that

$$H_\theta=F\left(t-\frac{z}{v},r\right)+G\left(t+\frac{z}{v},r\right)+f(r),$$

$$v_\theta=-\left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}F\left(t-\frac{z}{v},r\right)+\left(\frac{\mu}{\rho}\right)^{\frac{1}{2}}G\left(t+\frac{z}{v},r\right)+\varphi(r),$$

where $V=(\mu/\rho)^{\frac{1}{2}}H_0(r)$ and F, G, f, φ are arbitrary functions. *E. T. Copson* (St. Andrews).

McCluskey, E. J., Jr. Minimization of Boolean functions. Bell System Tech. J. **35** (1956), 1417–1444.

In the design of switching circuits [Shannon, same J. **28** (1949), 59–98; MR **10**, 671] for digital computers, telephone central offices, and digital machine tool controls, it is common practice to use boolean algebra. A systematic procedure is presented for writing a boolean function as a minimum sum of products. This leads to economy in the design of contact networks. The prodecure given here is a simplification and extension of the method presented by W. V. Quine [Amer. Math. Monthly **59** (1952), 521–531; **62** (1955), 627–631; MR **14**, 440; **17**, 814]. *S. Sherman* (Philadelphia, Pa.).

McCluskey, E. J., Jr. Detection of group invariance or total symmetry of a Boolean function. Bell System Tech. J. **35** (1956), 1445–1453.

For the purpose of minimization of boolean functions [see the preceding review] the author finds it convenient to determine from its normal form whether a boolean function has group invariance, i.e., whether there are any permutations or primings of the independent variable which leave the function unchanged. The method which he develops here for this purpose is extended to detecting whether such a function is equal to one whenever n, or more of its arguments, are equal to one. *S. Sherman.*

★**Gavrilov, M. A. Isolation of loops acting on a given element in relay circuits.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 5 pp.

Translated from Dokl. Akad. Nauk SSSR (N.S.). **87** (1952), 413–416.

★**Iablonskiĭ, S. V. Realization of linear functions in the Π-circuit.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 3 pp.

Translation from Dokl. Akad. Nauk SSSR (N.S.). **94** (1954), 805–806. The original Russian article was reviewed in MR **16**, 203.

★**Povarov, G. N. On the synthesis of multi-terminal switching networks.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 6 pp.

Translation from Dokl. Akad. Nauk SSSR (N.S.). **94** (1954), 1075–1078.

★**Ivanov, V. I. Cyclic relay circuits and analytic relations therein.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 5 pp.

Translated from Dokl. Akad. Nauk SSSR (N.S.). **104** (1955), 239–241. The original Russian article was reviewed in MR **17**, 1259.

★**Fabijanski, J. An attempt to apply Cauer's method to broad-band crystal filters.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 19 pp.

Translation from Pol. Inst. of Tech. **15** (1955), 12–18.

See also: Povarov, p. 555; Mikusinskiĭ, p. 575; Berk, p. 623; Shercliff, p. 619.

Thermodynamics and Heat

Romanov, A. G. Investigation of heat exchange in a dead-end channel in the case of natural convection. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1956**, no. 6, 63–76. (Russian)

The solution to the problem of the title is obtained by means of the methods of 1) similitude and dimensional analysis, and 2) integral equations of boundary-layer theory. The physical problem at hand satisfies a system of ten non-linear partial differential equations of the second order, on account of the fact that the fluid is viscous as well as compressible. At our present state of knowledge, however, these equations can not be solved to any reasonable degree of accuracy, and, therefore, the author resorts to various tolerable approximations. Unfortunately, however, the details of the analysis and the appropriate derivations are not presented (perhaps they can be found in a book (not available) by G. A. Ostroumov [Natural convection under conditions of an internal problem. Moscow, 1952]. The results of the experimental measurements seem to be in good agreement with the author's theory (not available). Comparisons with other experimental and theoretical investigations are also made. Complete details of the author's experimental set-up are given.

It is also stated that Lighthill's estimation of the coefficient of heat conduction is in error [Quart. J. Mech. Appl. Math. **6** (1953), 398–439]. *K. Bhagwandin.*

Šestopalov, V. P. General solution of the problem of the temperature boundary layer in a diffusor. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk 1956, no. 8, 3–9. (Russian)

The author investigates temperature distribution in a boundary layer along one wall of a diffusor. He considers two-dimensional convergent laminar flow of a viscous non-compressible liquid between two inclined walls. In Prikl. Mat. Meh. 16 (1952), 613–616 [MR 14, 697], the author found a particular integral of the non-homogeneous differential equation controlling the phenomenon; in this work he finds the complementary function, that is the solution of the homogeneous part of the above equation, together with the previously found particular integral gives the general solution. By suitable substitutions and changes of variables the author reduces the thermal equation into an ordinary hypergeometric differential equation obtaining a solution which contains, of course, hypergeometric functions. The solution enables the author to find an approximate expression for the heat transfer coefficient (the Nusselt number). *T. Leser.*

See also: Baratta, p. 580; Hellman, Habetler and Babrov, p. 602; Crank, p. 616.

Quantum Mechanics

Taniuti, Tosiya. On the theories of higher derivative and non-local couplings. II. Progr. Theoret. Phys. 15 (1956), 19–36.

A non-local interaction for a classical harmonic oscillator is considered, of the form

$$\ddot{q}+\omega^2 q=\int_{-\infty}^{\infty} f(t,t_1,\cdots,t_n)\times G(q(t),\dot{q}(t),q(t_1),\dot{q}(t_1),\cdots,q(t_n),\dot{q}(t_n))dt_1\cdots dt_n.$$

It is shown that under certain specified conditions the iterative solution constructed in a previous paper [same journal 13 (1955), 505–521; MR 17, 693] converges uniformly for all t, and that a unique solution exists satisfying boundary conditions for q and $\dot{q}$ at some specified time. The solution is equivalent to the general solution of a certain second order differential equation, which is obtained explicitly. The proof does not generalize to equations involving higher derivatives, except under very restrictive conditions.

For field theories with a similar type of non-local interaction the existence proof again holds. However it cannot be shown that the solution is equivalent to that of a second order differential equation, owing to the non-local character in a space-like direction, and causal relations are lost. *P. T. Matthews* (Rochester, N.Y.).

Arnous, Edmond. Théorie de l'effet Lamb-Retherford. Cahiers de Phys. no. 66 (1956), 1–5.

This is a review paper, presented at a conference, which makes no claim to contribute anything new. The Lamb shift is described in general physical terms, according to Welton's model. A useful table has been drawn up giving the contributions to the effect, for hydrogen and deuterium, of the various terms in powers of $Z\alpha$, the nuclear mass, electron magnetic moment, etc., with references to the corresponding calculations. *P. T. Matthews.*

Sherman, S. On Segal's postulates for general quantum mechanics. Ann. of Math. (2) 64 (1956), 593–601.

It is shown that the Jordan algebra $M_3{}^8$ of all 3×3 hermitian matrices with Cayley number coordinates satisfies the postulates of Segal [Ann. of Math. (2) 48 (1947), 930–948; MR 9, 241], with a suitable definition of the norm. The latter is derived from a partial ordering treated in a letter by A. A. Albert quoted in the paper, in which the non-negative elements are defined as squares and shown to be identical with the sums of squares. It thus appears that from a theoretical phenomenological point of view the algebra is entirely suited to represent observables, and especially such as might be related to the exceptional simple Lie groups, but there is no evidence that any physically known observables are described by it.

All previously known systems satisfying the postulates consisted of systems of operators on Hilbert space. The author shows that besides $M_3{}^8$, there exists a general class of systems not consisting of operators, and in fact forming non-distributive systems, satisfying the postulates. These systems do not behave like operator systems in certain formal respects, and are well suited to an examination of the independence of the postulates, one of which is shown to be redundant. *I. E. Segal.*

Cowan, R. D.; and Ashkin, J. Extension of the Thomas-Fermi-Dirac statistical theory of the atom to finite temperatures. Phys. Rev. (2) 105 (1957), 144–157.

The semiclassical statistical theory of the n-electron atom is extended to include the effects of exchange interactions and finite temperatures simultaneously. The starting points are (a) Fock's formula for the one-electron energy levels, the wave function being assumed to be an antisymmetrized product of one-electron functions, and (b) Fermi's formula for the statistical distribution of the electrons over the energy states. The one-electron space wave functions are approximated in the usual way by a semiclassical distribution in phase space and the one-electron energies are then evaluated from Fock's formula. A detailed discussion of the thermodynamic interpretation of the statistical distribution is given. *E. L. Hill* (Minneapolis, Minn.).

Stapp, Henry P. Relativistic theory of polarization phenomena. Phys. Rev. (2) 103 (1956), 425–434.

The paper works out in detail the formalism necessary to describe the scattering of polarized beams of spin one-half particles in relativistic quantum theory. It is possible in this case to label the initial and final states by a complete set of commuting observables for the particles (their momenta and a two-valued index describing the spin degree of freedom). However, it is more convenient for many purposes to introduce a redundant description by using the momentum and a four-valued spinor index. Then the scattering matrix has to satisfy the Dirac equation in each of the initial and final state spinor indices. This is the formalism the author develops. He considers the S-matrix for the scattering of spin one-half particles by spin zero and spin one-half particles. In both cases he succeeds in displaying it in a form very similar to that of the non-relativistic case. The changes necessary to pass from the non-relativistic to the relativistic case involve only the effect of Lorentz transformations on the four-vectors describing the polarization and momentum. *A. S. Wightman* (Copenhagen).

Davidson, P. M. Some theorems in group velocity. Proc. Edinburgh Math. Soc. (2) 9 (1956), 122–127.

The content of this paper is suggested in the author's words as follows: "in various types of dispersive media of

infinite extent the motion resulting from an arbitrary initial state of disturbance is characterized by certain quantities which are independent of time and may be expressed as suitable averages of the group velocity (or a power of it) over the range of waves present in a Fourier analysis of the original disturbance". The disturbances discussed are characterized by a function expressible either in the form

$$\int f(\nu, t) \exp i\nu x \, dx \text{ or } \int (f_- + f_+) \exp i\nu x \, dx,$$

where

$f = F(\nu) \exp -i\omega t$, $f_- = F_-(\nu) \exp -i\omega t$, $f_+ = F_+(\nu) \exp i\omega t$, and ω is a function of ν. By group velocity is meant $d\omega/d\nu$. *A. J. Coleman* (Toronto, Ont.).

Chang, T. S. Calculations of some operators in relativistic quantum mechanics. I, II. Acta Math. Sinica 3 (1953), 59–86. (Chinese. English summary)

Explicit calculations are made that lead to explicit expressions for the infinitesimal generators, of the inhomogeneous Lorentz group. *C. N. Yang.*

Ruijgrok, Th. W.; and Van Hove, L. Exactly renormalizable model in the quantum theory of fields. Physica 22 (1956), 880–886.

The model presented here is a generalization of the Lee model [Phys. Rev. (2) 95 (1954), 1329–1334; MR 16, 317]. Similar to that model, it also involves a scalar interaction of neutral bosons, θ, with infinitely heavy nucleons, V, but it permits not only two, but $n>1$ internal nucleon states V_q (q runs over all integers modulo n, $V_q = V_{q+n}$ for all q). The basic process is $V_q \rightleftarrows V_{q+1} + \theta$ and is associated with the coupling constant g_q^0. An essential weakness of the Lee model is thereby overcome, since the proposed model like any realistic field theory permits the successive emission and absorption of several bosons, which is not the case in the Lee model. Consequently, in this new model not only Z_2 is non-trivial, but also Z_1. The constant Z_3 is still trivially equal to one, since no pair production is possible. An exact expression for $Z_2(q)$ can be derived and one finds $Z_1(q) = Z_2(q+1)$, reminding one of Ward's identity. The renormalized coupling constants g_q are related to the g_q^0 by $g_q = Z_2^{1/2}(q) Z_1^{-1/2}(q) g_q^0$. In the limit of point nucleons (infinite cut-off) $g_q = (g_1^0 \cdots g_n^0)^{1/n}$, independent of q. Thus, this model includes as a special case not only the Lee model, but also the usual scalar interaction of neutral bosons with infinitely heavy point nucleons. In this limit the finite $\theta - V_q$ scattering cross section approaches zero, but a static $V_q - V_{q'}$ interaction remains finite even in the limit. Thus, the renormalized coupling constant is here measurable even for infinite cut-off. *F. Rohrlich.*

Ford, Kenneth W. Problem of ghost states in field theories. Phys. Rev. (2) 105 (1957), 320–327.

The question of the existence or non-existence of an anomalous state with negative probability in quantized field theories is discussed with the aid of an integral equation technique. The author considers this approach more general than the conventional, Hamiltonian treatment. To the author's surprise it turns out that the use of the integral equation does not allow one to change the physical content of the theory. No definite answer is given to the questions raised. *G. Källén.*

Fukuda, Nobuyuki; and Kovacs, Julius S. Integral equations for the transition matrices in the static meson theory. Phys. Rev. (2) 104 (1956), 1784–1790.

Chew and Low have derived an integral equation for the transition matrices in the static meson theory [Phys. Rev. (2) 101 (1956), 1570–1579]. This derivation is based on the one-meson states. Following similar arguments the present paper generalizes this equation for states involving an arbitrary number of mesons. Reducible and irreducible processes are not separated, which may be the cause of various ambiguities. Among the applications of this formalism, correction terms to the Chew-Low one-meson equation seem most interesting. No numerical work has been done. *F. Rohrlich.*

Bocchieri, P.; e Loinger, A. La condizione supplementare del campo di Stückelberg. Nuovo Cimento (10) 3 (1956), 626–632.

It is well-known that in quantum electrodynamics the difficulties connected with the existence of a supplementary condition can be overcome by the use of an indefinite metric in the abstract wave vector space (Gupta-Bleuler method).

The question whether the Gupta-Bleuler method can be generalized to other field theories with supplementary conditions is investigated by considering the most general field theory for a spin-one particle with non-vanishing mass (first proposed by Stuckelberg). It is shown that in this case also the use of the Gupta-Bleuler method allows for dealing with the supplementary condition in a consistent way.

From this result it is very plausible to conclude that a consistent treatment of all field theories with supplementary conditions can be obtained in a similar way. *S. Fubini* (Chicago, Ill.).

Polkinghorne, J. C. Causal products in quantum field theory. Proc. Cambridge Philos. Soc. 53 (1957), 261–262.

A new type of product of field operators is considered and its properties discussed. It is a function of a set of real numbers as well as of the operators, there being one number associated with each operator. This product satisfies certain causality conditions if the operators at space-like intervals commute; it is therefore proposed that it be called "causal product". *S. Deser.*

Preuss, H. Bemerkungen zum Self-consistent-field-Verfahren und zur Methode der Konfigurationenwechselwirkung in der Quantenchemie. Z. Naturf. 11a (1956), 823–831.

The self-consistent field method for molecules is discussed with a view towards its use with computing machines. Some preference is expressed for writing the zero order molecular functions as a sum of gaussians rather than a sum of exponentials or Slater functions. For the configuration interaction method it is suggested to approximate the molecular function by two-center functions (eigenfunctions of H_2^+). This assures the appearance of mostly tabulated integrals of the two-center and three-center type only. *F. Rohrlich.*

Ekstein, H. Scattering in field theory. Nuovo Cimento (10) 4 (1956), 1017–1058.

This paper contains a very detailed but somewhat formal discussion of the definition of asymptotic operators (in- and outgoing fields) in quantized field theories. The general discussion is illustrated with the aid of several examples, to a large extent taken from the domain of non-relativistic Schrödinger theory. The basic idea of the paper seems to be to apply a limiting prescription to

the energy momentum operators that guarantees the asymptotic states to have a physically reasonable energy momentum vector. The author states that "the problem consists in finding the asymptotic operators as functions of the bare particle operators". A formal system of equations for this purpose is given. *G. Källén.*

Green, Alex E. S. Approximate analytical wave functions for the nuclear independent-particle model. Phys. Rev. (2) **104** (1956), 1617–1624.

The bulk of this paper consists of a collection of diagrams which give some of the basic nuclear properties as calculated with a certain set of wave functions. The virtue of these wave functions is that they are very good approximations to the eigenfunctions of an independent particle Hamiltonian with a fairly realistic potential and that at the same time they can be expressed in terms of tabulated functions (Bessel functions and exponentials). The potential in point is of constant depth $-V_0$ out to a radius a, followed by an exponential tail characterized by the "width" d. This potential and the suitable determination of the parameters V_0, d, and $a=a_1A^{1/3}+a_0$ were discussed in a preceding series of papers [see ,e.g., A. E. S. Green and K. Lee, Phys. Rev. (2) **99** (1955), 772–777; MR **17**, 115]. The present paper gives as a function of the mass number: the normalization constants, the probabilities of finding a nucleon outside $r=a$, the expectation value of the radial part $\xi(r)$ of the spin-orbit interaction, the energy levels, and the radii (expectation values of r^2) for various orbitals. Considerations of the last item "leads one to expect a thin $(0-0.4\times 10^{-13}$ cm) neutron membrane around nuclei." *F. Rohrlich* (Iowa City, Iowa).

Ross, Marc. Pion effects on Fermi interactions. Phys. Rev. (2) **104** (1956), 1736–1741.

As the strong coupling between the nucleon and the pion became the center of much theoretical research in the past ten years, and as the various renormalization difficulties arose in field theories, the distinction between a "bare" particle and a "real" or "dressed" particle came more and more into the foreground. In particular, models of the nucleon have been proposed [e.g. R. G. Sachs, Phys. Rev. (2) **87** (1952), 1100–1110] in which the idea of a "bare" nucleon core and a pion "cloud" around the core is taken seriously. The present paper attempts to study the consequences of such a model on the problem of the universal Fermi interaction. It is assumed that it is the interaction of light fermions with bare nucleons which can be written in a simple form. The ensuing theory, while interesting in principle, does not solve any of the problems connected with the universal Fermi interaction. Moreover, the few phenomena which could test the validity of this theory would depend greatly on a specific model of the nucleon used in the evaluation.

M. J. Moravcsik (Upton, N.Y.).

Jones, D. S. On the scattering cross section of an obstacle. Phil. Mag. (7) **46** (1955), 957–962.

The theorem of van de Hulst [Physica **15** (1949), 740–746], giving the relation between the scattered amplitude and the sum of the scattering and absorption cross-sections for an obstacle in an incident plane wave, is derived by a calculation of the energy flow and by using a method that avoids certain convergence difficulties in van de Hulst's treatment [D. S. Jones, Proc. Cambridge Philos. Soc. **48** (1952), 733–741]. In addition it is shown that, when the incident wave is due to a point source, the sum of the two cross-sections is determined by the value of a certain field in the neighbourhood of the source. The argument could be applied to many scattering problems (e.g., sound waves or atomic collisions) but is given explicitly for the electromagnetic field.

C. J. Bouwkamp (Eindhoven).

Grosjean, C. C. A high accuracy approximation for solving multiple scattering problems in infinite homogeneous media. Nuovo Cimento (10) **3** (1956), 1262–1275.

An approximate formula is obtained for the steady-state density of isotropically scattered particles emitted by a point source in an infinite homogeneous medium. The author's approximation is claimed to be more simple, at the same time giving better results, than all previous approximations [e.g., P. I. Richards, Phys. Rev. (2) **100** (1955), 517–522]. In deriving his formula, the author starts from an approximation obtained in an earlier paper [Nuovo Cimento (9) **11** (1954), 11–40]. Numerical evidence supporting the author's claim is displayed in a number of tables and curves. Results for media of finite extent are promised in a later paper. *C. J. Bouwkamp.*

See also: Penzlin, p. 584; Temperley, p. 610.

Relativity

Shibata, Takashi. The "Lorentz transformations without rotation" and the new fundamental group of transformations in special relativity and quantum mechanics. J. Sci. Hiroshima Univ. Ser. A. **19** (1955), 101–112.

The author continues his investigations [same J. **16**, (1952), 61–66, 285–290, 487–496 (1953); **17** (1953), 67–73; MR **15**, 752] of the 3-parameter sub-group G_3, of the Lorentz group, which leaves invariant a given null-ray. By relating his transformations to the corresponding "Lorentz transformations without rotation", he derives the Thomas-like precession induced by their successive application. The equations of the group G_3 are expressed in an appropriate 4-dimensional notation [but the four generators defined by Eq. (3.1) for $\mu=1, 2, 3, 4$ are not linearly independent; G_3 can, e.g., be generated by the three generators $\mu=1, 2, 3$]. *H. P. Robertson.*

Arnowitt, R. L. Phenomenological approach to a unified field theory. Phys. Rev. (2) **105** (1957), 735–742.

This is a most interesting and well founded attempt to include geometrically the electromagnetic field into the frame of general relativity. The basic tools are the gravitational symmetric tensor field $g_{\lambda\mu}$, the electromagnetic skew symmetric tensor field $f_{\lambda\mu}$ given by means of the world vector Q_μ, $f_{\mu\lambda}=2\partial_{[\mu}Q_{\lambda]}$ a symmetric connection $\Lambda_{\lambda\mu}{}^\nu$ (to be defined later) and a non-symmetric connection

$$\Gamma_{\lambda\mu}{}^\nu=\Lambda_{\lambda\mu}{}^\nu+\delta_\lambda{}^\nu Q_\mu$$

[this equation excludes for $Q\neq 0$ the Einstein unified theory; cf. Hlavatý, J. Rational Mech. Anal. **3** (1954), 103–146; MR **15**, 654].

Conformal change of the metric $g_{\lambda\mu}\to(\exp 2\Lambda)g_{\lambda\mu}$ is always coupled with the change $Q_\mu\to Q_\mu+\partial_\mu\Lambda$ so that $f_{\mu\lambda}$ remains invariant. If $R_{\mu\lambda}$ $(W_{\mu\lambda})$ denotes the contracted curvature tensor of $\Gamma_{\lambda\mu}{}^\nu$ $(\Lambda_{\lambda\mu}{}^\nu)$ then obviously

$$R_{\mu\lambda}=W_{\mu\lambda}+f_{\lambda\mu}. \tag{1}$$

The Lagrangian $\mathfrak{L}$ of this theory is defined by

$$2\mathfrak{L}=|\mathfrak{g}|^{-\frac{1}{2}}[\alpha_1 R_{\alpha\beta}R^{\alpha\beta}+\alpha_2(R_\alpha{}^\alpha)^2].$$

Here $\mathfrak{g}\overset{\text{def}}{=}\operatorname{Det}((g_{\lambda\mu}))$, α_1, $\alpha_2=\text{const}$ and the indices are raised by means of $g^{\lambda\nu}\overset{\text{def}}{=}\partial \ln \mathfrak{g}/\partial g_{\lambda\nu}$. The variation of $L=\int d^4\mathfrak{L}$ with respect to Λ and Q leads to the field equations

$$\text{(2)a)}\qquad \dot{\nabla}_\lambda \mathfrak{G}^{(\mu\nu)}=\delta^{(\mu}{}_\lambda\dot{\nabla}_\beta\mathfrak{G}^{\nu)\beta}$$

$$\text{b)}\qquad \partial_\lambda\mathfrak{G}^{[\lambda\nu]}=0.$$

Here $\dot{\nabla}$ refers to Λ and

$$\text{(3)}\qquad \mathfrak{G}^{\lambda\nu}\overset{\text{def}}{=}\frac{\delta\mathfrak{L}}{\delta R_{\lambda\nu}}=|\mathfrak{g}|^{-\frac{1}{2}}(\alpha_1 R^{\lambda\nu}+\alpha_2R_\alpha{}^\alpha g^{\lambda\nu})$$

so that $\mathfrak{G}^{\lambda\nu}$ is conformally invariant. Putting

$$^*g^{\mu\nu}=\mathfrak{G}^{(\mu\nu)}|^*\mathfrak{g}|^{\frac{1}{2}}$$

$$(^*\mathfrak{g}=\operatorname{Det}((^*g_{\lambda\mu})),\ ^*g_{\lambda\mu}=\partial\ln {}^*\mathfrak{g}/\partial{}^*g^{\lambda\mu})$$

one obtains immediately from (2)

$$\text{(4)}\qquad \Lambda_{\lambda\mu}{}^\nu=\tfrac{1}{2}{}^*g^{\nu\alpha}(\partial_\lambda{}^*g_{\alpha\mu}+\partial_\mu{}^*g_{\alpha\lambda}-\partial_\alpha{}^*g_{\lambda\mu})$$

while according to (1) and (3) the equation (2)b) is equivalent to the second set of source-free Maxwell's equations.

The variation of $\mathfrak{L}$ with respect to $g^{\mu\nu}$ yields the second set of field equations.

$$\text{(5)}\qquad \alpha_1(W_\mu{}^\alpha W_\alpha{}^\nu-\tfrac{1}{4}\delta_\mu{}^\nu W_\alpha{}^\beta W_\beta{}^\alpha)+$$
$$+\alpha_2(W_\mu{}^\nu W_\alpha{}^\alpha-\tfrac{1}{4}\delta_\mu{}^\nu(W_\alpha{}^\alpha)^2)=\alpha_1T_\mu{}^\nu$$

where $T_\mu{}^\nu$ is the usual electromagnetic stress energy tensor. — These are some of the main features of the paper under consideration, which contains also the spherically symmetric solution.

The choice of the objects referred to in 1) as well as of the Lagrangian $\mathfrak{L}$ is founded on four assumptions which constitute the phenomenological approach to the problem.

V. Hlavatý (Bloomington, Ind.).

See also: Manarini, p. 622; Chang, p. 626; García, p. 628.

Astronomy

Contopoulos, George. On the motions of stars in an ellipsoidal stellar system. Astrophys. J. **124** (1956), 643–651.

If the surfaces of constant stellar density are similar ellipsoids, this density is a function of the similarity parameter. An infinite series for the density in terms of the similarity parameter is assumed and the gravitational potential is worked out. The equations of motion for an individual star are deduced and solved in series by the method of undetermined coefficients and also by that of approximate osculating orbits. Perturbations are also briefly considered. *G. C. McVittie* (Urbana, Ill.).

Przybylski, Antoni. A variational method for improving model stellar atmospheres. Monthly Not. Roy. Astr. Soc. **115** (1955), 650–660 (1956).

The temperature distribution in a non-gray atmosphere in radiative equilibrium is determined by the condition that the integrated net flux, $\int_0^\infty F_\nu(t)d\nu$ is a constant where

$$F_\nu(t)=\int_0^\infty B_\nu(T_\tau)E_2(\lambda_\nu|\tau-t|)d(\lambda_\nu t)$$

and λ_ν is some prescribed function of ν and τ, E_2 is the second exponential integral and $B_\nu(T_\tau)$ is the Planck function for the temperature T_τ which prevails at τ. The method of solution generally adopted is to use for T_τ the explicitly known solution for the gray case (i.e. $\lambda_\nu=1$) and "correct" it by a process essentially of trial and error. In practise this process is carried out in one of many ways. {In the present paper the author describes one further numerical process of obtaining the solution; though the reviewer cannot see why it is called a "variational method": it is very much more a "relaxation" method.} *S. Chandrasekhar* (Williams Bay, Wis.).

Lenoble, J. Calcul du rayonnement diffusé dans une couche de brume. J. Sci. Météorol. **8** (1956), 23–28. (Spanish summary)

By methods described in an earlier paper [Rev. Opt. **35** (1956), 1–17; MR **17**, 1256], the author obtains solutions for the intensity of radiation at various points in the interior of a finite layer of dense mist when (1) illuminated by a uniform sky and (2) by a plane-parallel beam incident at an angle of 60°. The phase function considered is

$$p(\cos\Theta)=\frac{1}{4\pi}[1+1.73P_1(\cos\Theta)+P_2(\cos\Theta)].$$

The solutions are tabulated and the results of the calculations are illustrated by graphs for certain special cases.

S. Chandrasekhar (Williams Bay, Wis.).

Simon, R. Etude de la propagation des ondes dans le modèle de Roche généralisé. Ann. Astrophys. **19** (1956), 115–121. (English and Russian summaries)

The small radial oscillations of a generalized Roche model gaseous envelope surrounding a spherical central body have been shown to depend on Hankel functions of the first kind and of order ν. But the condition that the total mass of the envelope be finite imposes the condition that the ratio of specific heats, γ, shall be less than $64/(49-9\nu^2)$. It is shown that $\nu=3/2$, which is compatible with this condition for $\gamma=5/3$, leads to an analytical solution of the problem. If the maintenance of the oscillations is regarded as due to emission of energy by the central body, it is possible to calculate an upper limit to the emission compatible with the linearized theory that is being used. This energy emission is much larger than that normally accepted in stellar interiors.

G. C. McVittie (Urbana, Ill.).

García, Godofredo. The equations of the gravitational and cosmological fields. Contracting universe-expanding universe. Actas Acad. Ci. Lima **18** (1955), 3–82. (Spanish)

The author proposes the metric

$$ds^2=V^2(q)dt^2-G^2(t)\sum_1^3 a_{ij}(q)dq^idq^j,$$

where the $a_{ij}(q)$ define a 3-space of constant curvature, as the most general required to describe both static gravitational fields and those of relativistic cosmology. He then computes curvature tensors and field equations for the general field, and discusses special cases.

H. P. Robertson.

Geophysics

Mejzlík, Ladislav. Die Anwendung der Methode der Netze zur Lösung von Problemen der Grundwasserströmung unter Wasserbauwerken. Apl. Mat. 1 (1956), 399–430. (Czech. Russian and German summaries)

The author considers plane problems of laminar filtration of underground water under waterworks. He assumes non-compressible liquid of constant viscosity and saturation of pores. He derives the partial differential equations controlling the phenomenon for homogeneous and non-homogeneous, isotropic and anisotropic regions. The differential equations are approximated by difference equations at orthogonal grids. The author discusses different methods of solutions at various boundary conditions, analyzes mesh size and lists numerical methods for simultaneous systems in case of implicit scheme.

T. Leser (Aberdeen, Md.).

Ertel, Hans. Eine Kompatibilitäts-Bedingung der höheren Geodäsie. S.-B. Deutsch. Akad. Wiss. Berlin. Kl. Math. Phys. Tech. 1956, no. 4, 14 pp.

The "compatibility-condition" is equivalent to the extended form of Clairaut's theorem as found in Jordan and Eggert [Handbuch der Vermessungskunde, vol. 3, part 2, 9th ed., Metzler, Stuttgart, 1948, pp. 299–300]. The derivation differs in detail but is the same in essence. The author makes explicit an assumption which is implied by Jordan and Eggert [loc. cit.] — namely, that the various boundary conditions of the earth's gravitational field must be compatible with one another. *B. Chovitz.*

See also: De, p. 617; Dorrestein, p. 617.

OTHER APPLICATIONS

Games, Economics

Bottema, O. The Shylock game. Nieuw Arch. Wisk. (3) 4 (1956), 127–131.

The author gives the complete theory of the gambling game about which Herbert Adams wrote "The old jew mystery" [Collins, London, 1936]. It is played with dice and counters by any number of players. Certain numbers of counters are paid into or taken out of twelve compartments according to prescribed rules. The mathematical expectation of each player is expressed as a series, and computed to five significant figures in some typical cases.

H. S. M. Coxeter (Toronto, Ont.).

★Kuhn, H. W. On a theorem of Wald. Linear inequalities and related systems, pp. 265–273. Annals of Mathematics Studies, no. 38. Princeton University Press, Princeton, N. J., 1956. $5.00.

In 1934 Wald gave the first rigorous proof of the existence of a solution to the Walras-Cassel equations of static general equilibrium, in the case where the technology is of fixed-coefficient type. His results have since been extended by McKenzie, Arrow-Debreu, and others. Given an $m\times n$ matrix a of non-negative elements, an m-vector r of positive elements, and demand functions $\sigma_j=f_j(s)$, a solution consists of nonn-egative n-vectors s and σ, and a non-negative m-vector ρ such that a) $as\leq r$; b) if strict inequality holds in any component, the corresponding component of ρ is zero; c) $a'\rho=\sigma$; and d) $\sigma_j=f_j(s)$. Conditions must be placed on a and the mapping f.

A very perspicuous proof of Wald's Theorem is here given, using as tools the duality theorem of linear programming and the fixed-point theorem of Kakutani. In the last section it is remarked that much of the remaining difficulty stems from Wald's requirement that in the solution, $s>0$, and the rather odd condition that he placed on the (inverse) demand functions to insure this. If this economically unsatisfying restriction is removed, the proof is further simplified. *R. Solow.*

Beckerman, W. The world trade multiplier and the stability of world trade, 1938 to 1953. Econometrica 24 (1956), 239–252.

A Leontief model is used to describe the relation between changes in imports from the rest of the world, sub-divided into nine trading areas, and consequent changes in total world trade. The basic assumption is that imports of a trading area from another one (excluding the U.S.A.) depend linearly on total exports of that area, i.e. $M\mathbf{x}+\lambda\mathbf{a}=\mathbf{x}$, where M is a 9 by 9 matrix of import coefficients, $\mathbf{x}$ is a vector of area exports, λ a scalar denoting total U.S. imports with $\mathbf{a}$ their distribution over the areas. The effect of domestic income on imports and exports is ignored throughout.

The "world trade multiplier" is then obtained as the sum of $(I-M)^{-1}\mathbf{a}$. This total denotes the change in world trade arising from a unit change in U.S. imports. The multiplier is shown to have declined from 8.48 in 1938 to 6.14 in 1948 to 5.52 in 1953, due presumably to increased exports from the U.S.A. to the rest of the world.

G. Morton (Raleigh, N.C.).

Allais, M. Explication des cycles économiques par un modèle non linéaire à régulation retardée. Metroecon. 8 (1956), 4–83.

A theory is suggested for explaining economic cycles by means of the interplay of the credit policy and the propensity to liquidity. The theory is intended to be explanatory only and is not capable of providing predictions because the propensity to liquidity is psychological and is not under governmental control. It is claimed that the theory is economically simple, although it has functional symbols as parameters, but it is mathematically complicated to work out the consequences other than approximately. Numerical comparisons are made with published economic data. *I. J. Good* (Cheltenham).

Seng, You Poh. Some theory of index numbers. I. J. Roy. Statist. Soc. Ser. A. 119 (1956), 312–332.

The author's aim is to set up the results of an investigation into desirable properties of an index number other than those already expressed by I. Fisher in his treatise "The Making of Index Numbers". These properties are maintenance of continuity when 1) the base year is changed, 2) one group of new items is substituted instead of old items, 3) new items are added to the list, 4) items are deleted, 5) weighting system is changed, or 6) when many changes of this kind take place simultaneously. He tries to appraise different methods from this point of view without reaching any decidedly demonstrated new results.

L. Törnqvist (Helsinki).

Banerjee, K. S. A note on the optimal allocation of consumption items in the construction of a cost of living index. Econometrica 24 (1956), 294–295.

In cost of living index numbers, price data are normally compiled for only a few of those commodities which are actually represented by the aggregate items. The author suggests that the commodities should be chosen in accordance with Neyman's method for optimal allocation of the number of items to be observed [J. Roy. Statist. Soc. 97 (1934), 558–606]. *L. Törnqvist* (Helsinki).

Tintner, Gerhard. Complementarity and shifts in demand. Metroecon. 4 (1952), 1–4.

Complementarity in the sense of O. Lange [Rev. Econ. Studies 8 (1940), 58–63] is analysed in terms of the derivatives of the utility function with respect to advertising.
H. Wold (Uppsala).

Krumbach, Günther. Das Toto-Roulettespiel der Saarland-Sporttoto-G. m. b. H., seine Theorie und ein Vergleich mit den tatsächlichen Ergebnissen. Ann. Univ. Sarav. 5 (1956), 228–234 (1957).

Rios, Sixto. Methods and problems of Operations Analysis. Trabajos Estadist. 7 (1956), 187–198. (Spanish)
An expository lecture.

Johnson, Gordon K.; and Turner, Inez M. Use of transfer functions for company planning. Operations Res. 4 (1956), 705–710 (1957).

San Juan Llosá, Ricardo. The "simplex" method in linear programming. Trabajos Estadist. 7 (1956), 199–219. (Spanish)

Readers who are familiar with expositions of the "simplex" method will find nothing new here, except a simplification in determining the maximum.
Author's note.

See also: Bellman, p. 582; Des Raj, p. 606; De, p. 617.

Biology and Sociology

Li, C. C. The concept of path coefficient and its impact on population genetics. Biometrics 12 (1956), 190–210.

A path coefficient is the same as a regression coefficient when the variables are measured in units of standard deviations. The method of path coefficients is a technique specially adapted to correlation problems of the kinds occurring in genetics. It was introduced by S. Wright [J. Agric. Res. 20 (1921), 557–585] and subsequently extensively developed, mainly by him, in a long series of papers. The main references are listed in the paper under review, which gives a very clear account of the method and its simpler applications.
I. M. H. Etherington (Edinburgh).

See also: Bliss, p. 609.

Information and Communication Theory

★**Khinchin, A. Ia. On the fundamental theorems of information theory.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 84 pp.

Translated from Uspehi Mat. Nauk (N.S.) 11 (1956), no. 1(67), 17–75. The original Russian article was reviewed in MR 17, 1098.

Control Systems

★**Shastova, G. A. Investigation of noise-stability of remote control command transmission by the methods of potential noise-stability theory. I.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 19 pp.

Translation from Avtomatika i Telemehanika 16 (1955), 344–355.

★**Siforov, V. I. On noise-stability of a system with correcting codes.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 16 pp.

Translation from Radiotekh. i Elektr. 1 (1956), 131–142.

★**Shestakov, V. I. Algebraic method of designing sequential relay.** Translated by Morris D. Friedman, 572 California St., Newtonville 60, Mass., 1956. 6 pp.

Translation from Dokl. Akad. Nauk SSSR (N.S.) 99 (1954), 987–990. The original Russian article was reviewed in MR 16, 786.

See also: Bellman, p. 582.

HISTORY, BIOGRAPHY

Clagett, Marshall. The Liber de motu of Gerard of Brussels and the origins of kinematics in the West. Osiris 12 (1956), 73–175.

The treatise has four parts. The first part discusses ancient kinematics and the extent to which Latin translations were available to Gerard; the second deals with questions of authorship; the third gives a critical text of the Liber de motu, while the fourth analyzes the contents of the Liber.

Baron, Roger. Hugonis de Sancto Victore, Practica Geometriae. Osiris 12 (1956), 176–224.

Earlier scholars assigned this treatise to an otherwise unknown Hugo. Arguments are now given for assigning it to Hugues de Saint-Victor. A description is given of the extant manuscripts.

Amir-Moéz, A. R. Ibn Haitham's problems and their geometric solutions. Math. Mag. 30 (1956), 93.

The problems are: i) find the direction of a ray of light through a point P interior to a circle such that after two reflections from the circle the ray passes again through P; ii) find the ray through a given point A whose reflection passes through a given point B. Ibu Haitham (965–1039) set up these problems as a quadratic and a biquadratic equation respectively but was dissatisfied because he could not find "geometrical" solutions. The present article provides a suitable ruler-and-compass solution for the first of the two problems.

nique
kinds
right
y ex-
es of
paper
f the

h).

ns of
Fried-
1956.

1956),
iewed

ty of
ethods
ed by
lle 60,

a 16

h cor-
lman,
16 pp.
131-

quen-
, 572
o.
N.S.).
as re-

ven of

their

f light
er two
gh P;
ection
-1039)
dratic
use he
present
ion for